EPFL

Politecnico
di Torino
Dipartimento
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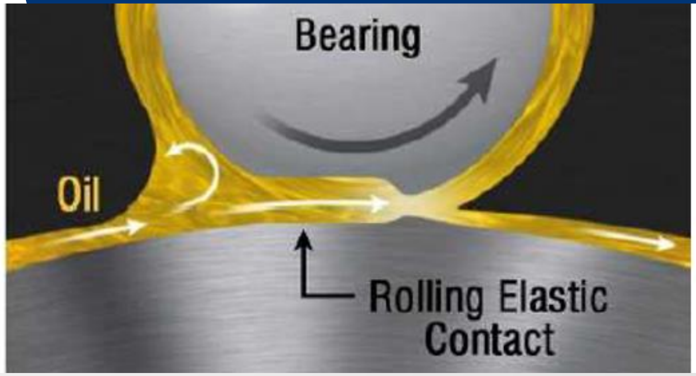
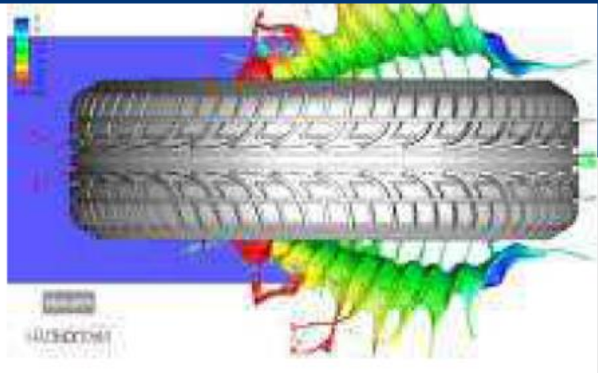




THIN FILM LUBRICATION AND GAS-LUBRICATED BEARINGS

THIRD LEVEL COURSE

PROF. J. SCHIFFMANN, PROF. F. COLOMBO

OUTLINE

Thin film lubrication

1. Intro and derivation of the **Reynolds equation**

2. Analytical & numerical solution of simple cases
    - 1D hydrodynamic slider
    - 1D slider with normal squeeze motion
    - Lubricated cylinder on plane
    - Journal bearing with infinite length
    - 2D hydrodynamic slider

Federico
Colombo

Luigi
Lentini

Edoardo
Goti

from Politecnico di Torino

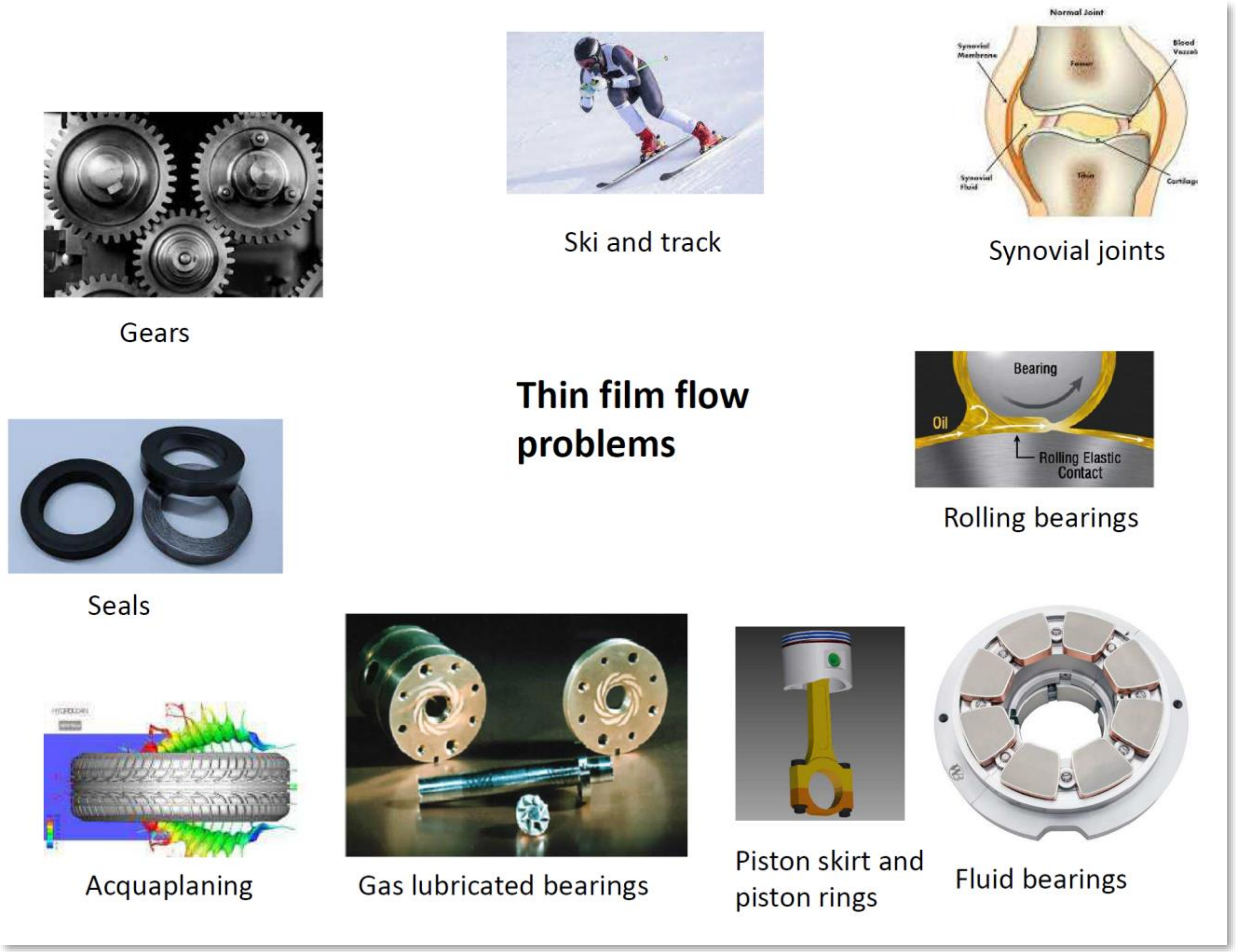
INTRODUCTION ON LUBRICATION

Lubrication is often a need to **save energy** (reduce losses due to friction) and to **increase life** of the components in contact through a surface and in relative motion (reduce or **prevent wear**).

The behavior of sliding surfaces is strongly modified with the introduction of a lubricant between them:

- Solid
- Liquid
- gaseous lubricant

A very important parameter of the lubricant is the **viscosity**, to which friction is proportional.



LUBRICATION REGIMES

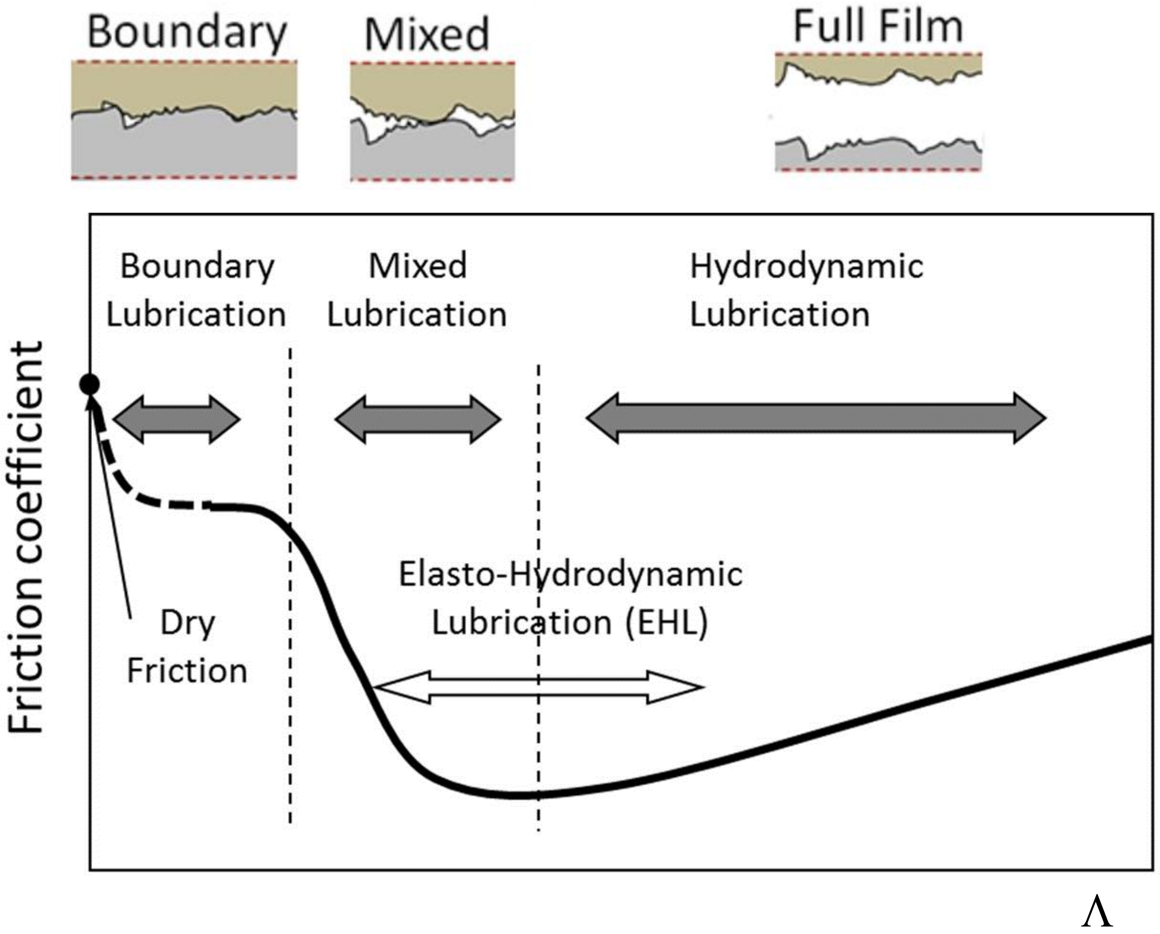
Λ FACTOR

Dimensionless film height between two rough surfaces *a* and *b*:

$$\Lambda = \frac{h_{min}}{\sqrt{R_{qa}^2 + R_{qb}^2}}$$

Lubrication regimes:

$\Lambda < 1$	boundary lubrication
$1 < \Lambda < 3$	mixed lubrication
$\Lambda > 3$	full film lubrication



LUBRICATION REGIMES

ROUGHNESS CALCULATION

Average roughness:

$$R_a = \frac{1}{L}\int_0^L |y(x)|dx$$

Standard Deviation roughness
or *root mean square (RMS) roughness:*

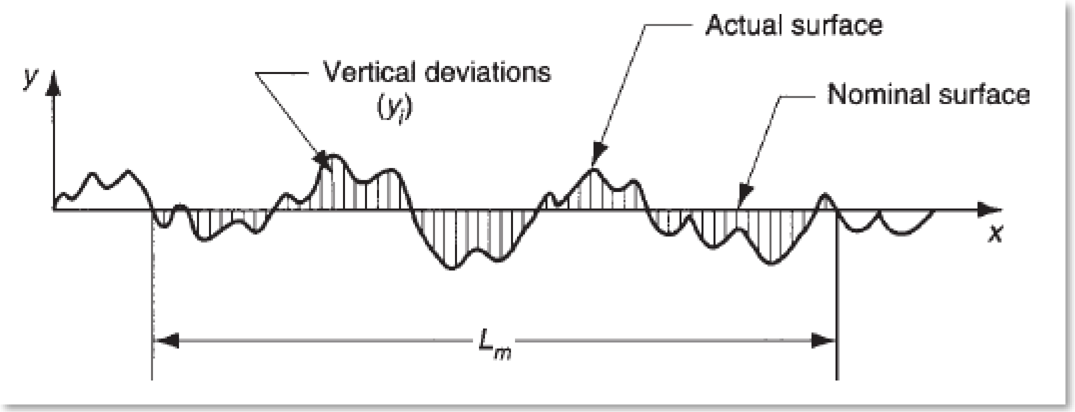
$$R_q = \sqrt{\frac{1}{L}\int_0^L y(x)^2 dx}$$

Peak-to-valley roughness:

$$Rt = \max(y) - \min(y)$$

L: sample length L

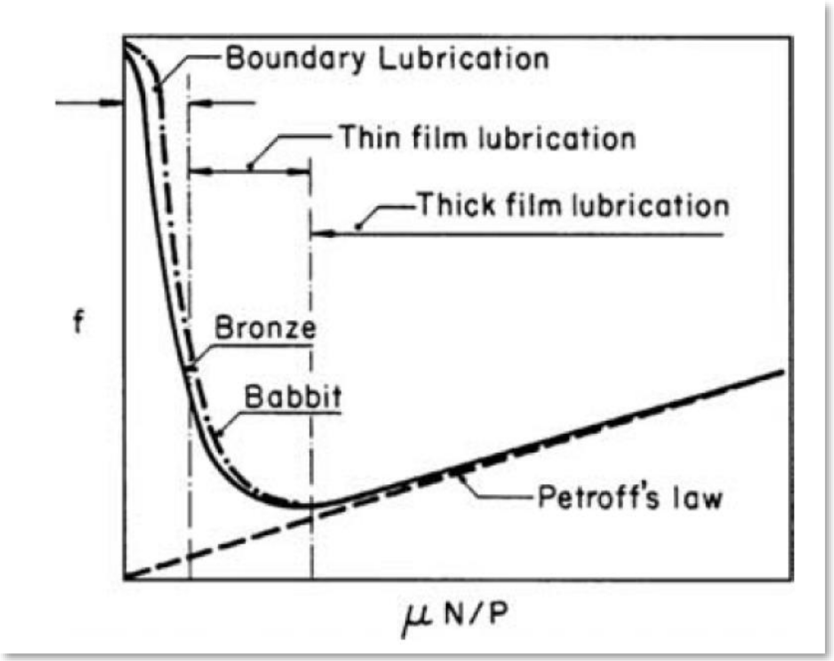
$y(x)$: surface deviation from the reference line (nominal surface)



LUBRICATION REGIMES

DIMENSIONLESS NUMBER

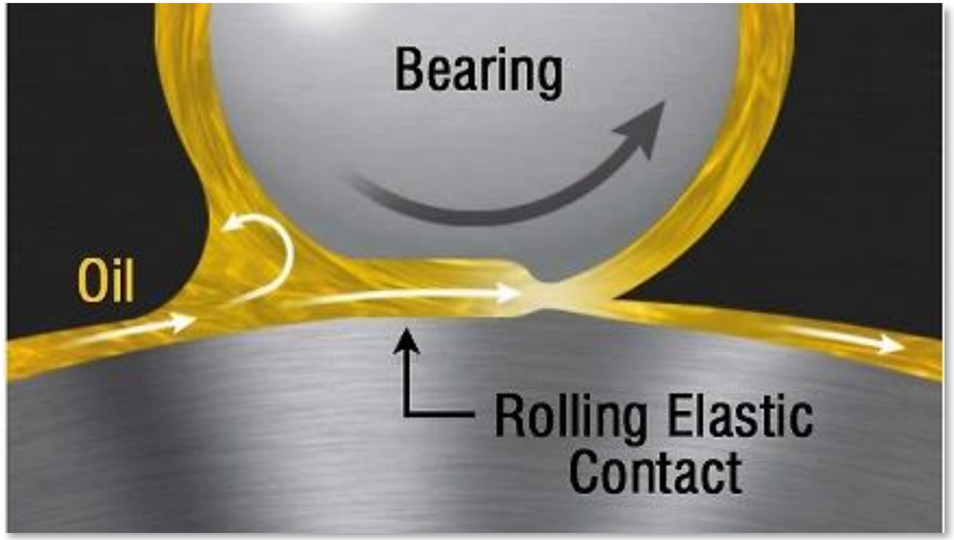
Sommerfeld number: $S = \mu N / P$
N: relative speed
P: load.



Low speeds and high loads → Low Sommerfeld number → high coefficient of friction (boundary lubrication)

High speeds and low loads → High Sommerfeld number → the c.o.f. comes back to increase (full film lubrication)

Analytical derivation of Reynolds equation



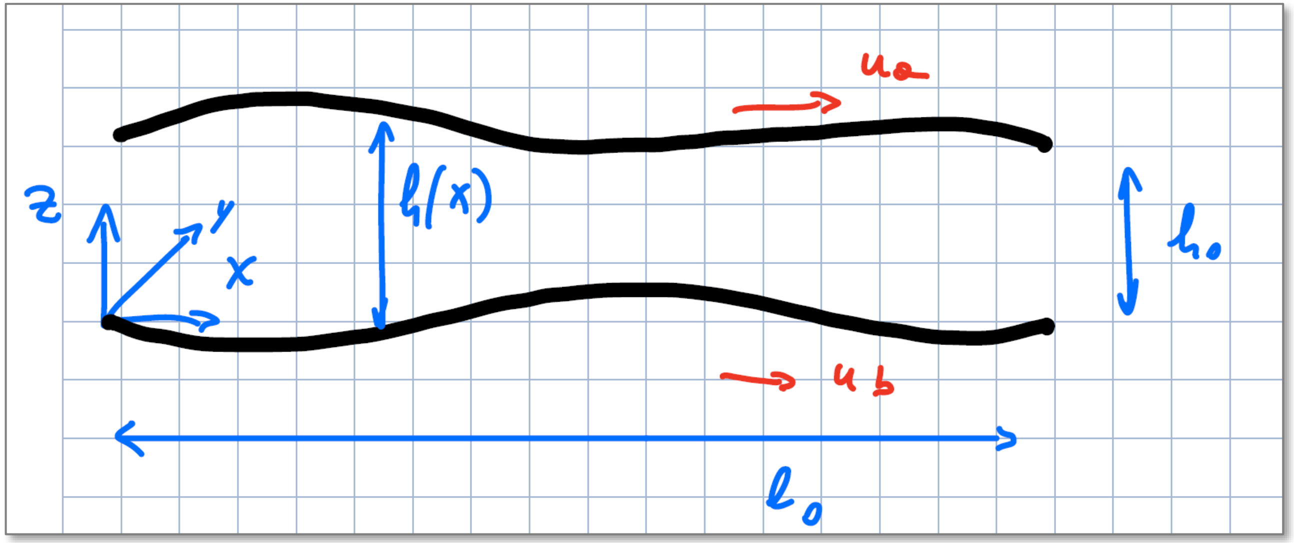
1D REYNOLDS’ EQUATION

Let us consider two rigid surfaces separated by a lubricant fluid.

$h(x)$: clearance
h_0: mean clearance
l_0: characteristic length

Where:

u: speed of lubricant along x direction
v: speed of lubricant along y direction
w: speed of lubricant along z direction



Hypotheses:

1. The flow is only along the x-direction, i.e. speed $v = 0$, $w \to 0$ (negligible)
2. Flow is laminar
3. Inertia forces are negligible with respect to viscous forces
4. Speed *u* is independent of *y*-coordinate (1D)

1D REYNOLDS' EQUATION

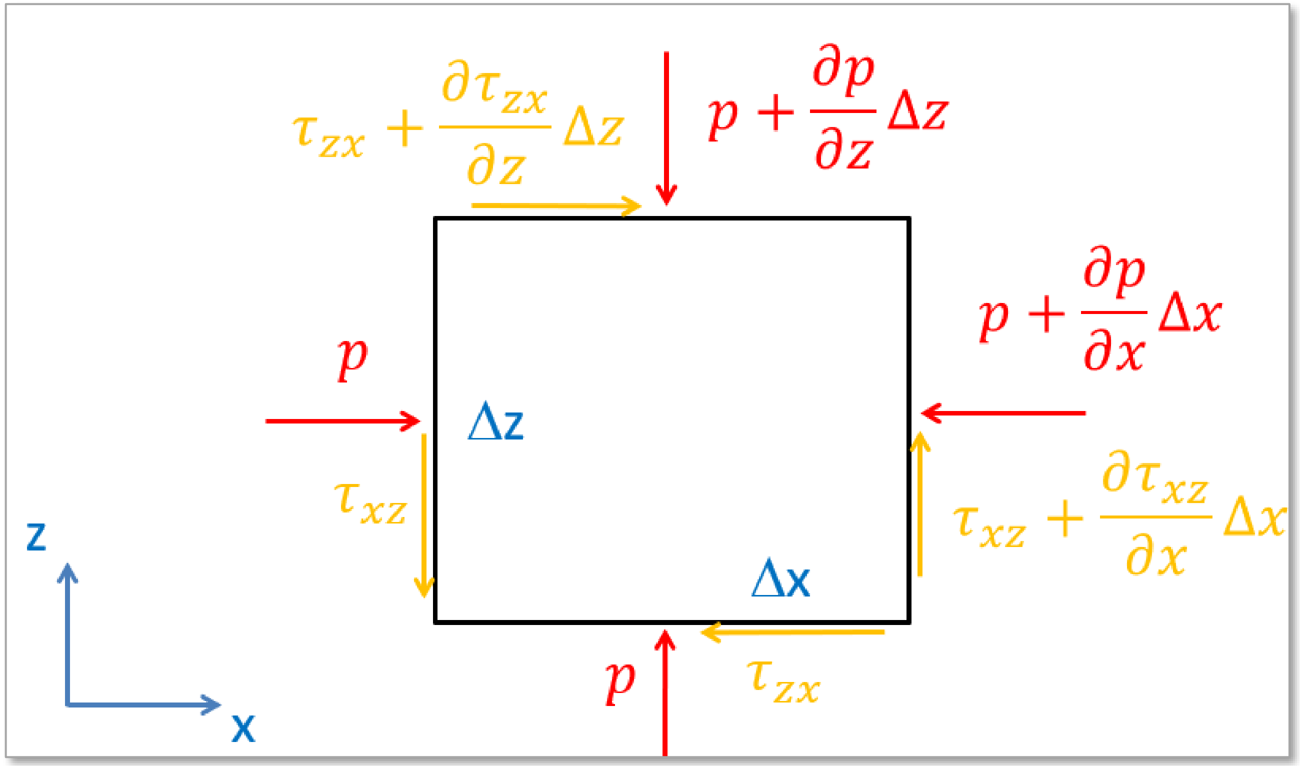
MOMENTUM EQUATIONS OF THE FLUID ELEMENT

Equilibrium along x: $$\frac{\partial p}{\partial x}\Delta x\,\Delta z = \frac{\partial \tau_{zx}}{\partial z}\Delta z\,\Delta x$$

Equilibrium along z: $$\frac{\partial p}{\partial z}\Delta z\,\Delta x = \frac{\partial \tau_{xz}}{\partial x}\Delta x\,\Delta z$$

Rotation in plane xz: $$\tau_{zx}\Delta x\,\Delta z = \tau_{xz}\,\Delta z\Delta x$$

$$\begin{cases} \dfrac{\partial p}{\partial x} = \dfrac{\partial \tau_{zx}}{\partial z} & (1) \\[2ex] \dfrac{\partial p}{\partial z} = \dfrac{\partial \tau_{xz}}{\partial x} & (2) \\[2ex] \tau_{zx} = \tau_{xz} & (3) \end{cases}$$



1D REYNOLDS’ EQUATION

SHEAR STRESS VS SHEAR RATE

- The behaviour of solids is (shear) **strain**-dependent

Elastic material

$$\sigma = E\varepsilon$$

$$\tau = G\gamma \qquad \gamma_{zx} = \frac{\partial x}{\partial z} + \frac{\partial z}{\partial x}$$

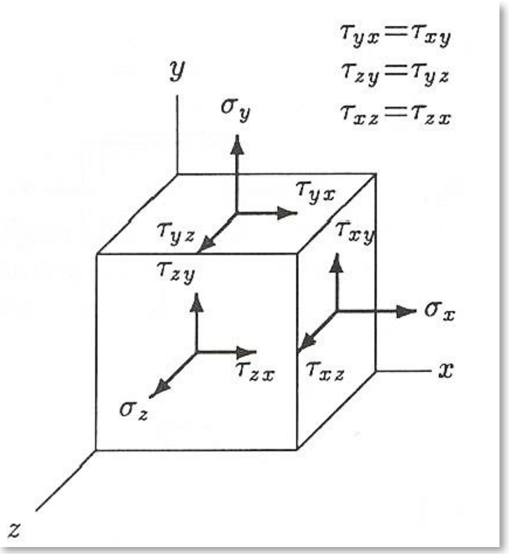
- The behaviour of liquids is **shear rate**-dependent

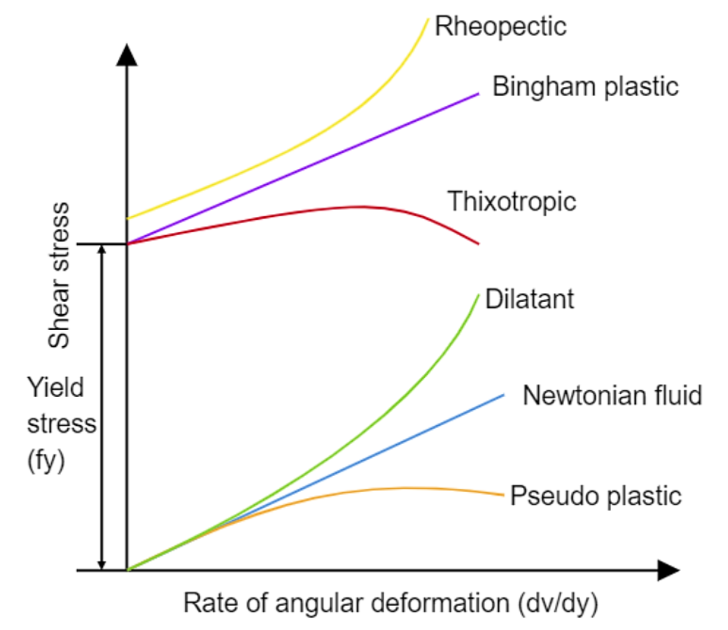
Newtonian fluid

$$\tau = \mu\dot{\gamma}$$

$$\dot{\gamma}_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$





NB: μ has constant value in space and time for Newtonian fluids

1D REYNOLDS' EQUATION

Hypotheses:

5. Thin film $\frac{h_0}{l_0} \ll 1, \qquad \left(\frac{\partial w}{\partial x}, \frac{\partial u}{\partial x}\right) \ll \frac{\partial u}{\partial z}$

6. Viscous, Newtonian fluid $\tau_{zx} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \cancel{\frac{\partial w}{\partial x}}\right) \cong \mu\,\frac{\partial u}{\partial z}$

Equation (1) turns into $\frac{\partial p}{\partial x} = \frac{\partial \tau_{zx}}{\partial z}$ ➡ $\boxed{\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial z^2}}$

Integrating along z: $\int \frac{\partial^2 u}{\partial z^2} dz = \frac{1}{\mu}\int \frac{\partial p}{\partial x} dz \qquad u = \frac{1}{2\mu}\frac{\partial p}{\partial x}z^2 + c_1 z + c_2$

Equation (2) turns into $\frac{\partial p}{\partial z} = \frac{\partial \tau_{xz}}{\partial x} = \frac{\partial}{\partial x}\left(\mu\,\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial z}\left(\mu\,\frac{\partial \cancel{u}}{\partial x}\right) \to 0$

***p* does not vary across the z-direction**

$$\boxed{p = p(x)}$$

1D REYNOLDS' EQUATION

BOUNDARY CONDITIONS

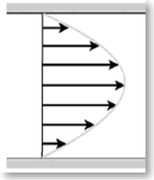
Boundary conditions for the speed:

$$\begin{cases} u(z=h) = u_\mathrm{a} \\ u(z=0) = u_\mathrm{b} \end{cases} \qquad \begin{cases} w(z=h) = w_\mathrm{a} \\ w(z=0) = w_\mathrm{b} \end{cases}$$

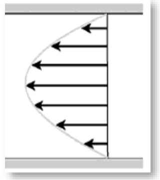
Constants c_1 and c_2 are determined:

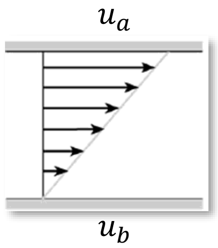
$$u = \underbrace{\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(z^2 - zh\right)}_{\text{Parabolic (Poiseuille) term (pressure effect)}} + \underbrace{u_a\frac{z}{h} + u_b\left(1 - \frac{z}{h}\right)}_{\text{Linear (Couette) term (dragging effect)}}$$

$\dfrac{\partial p}{\partial x} < 0$



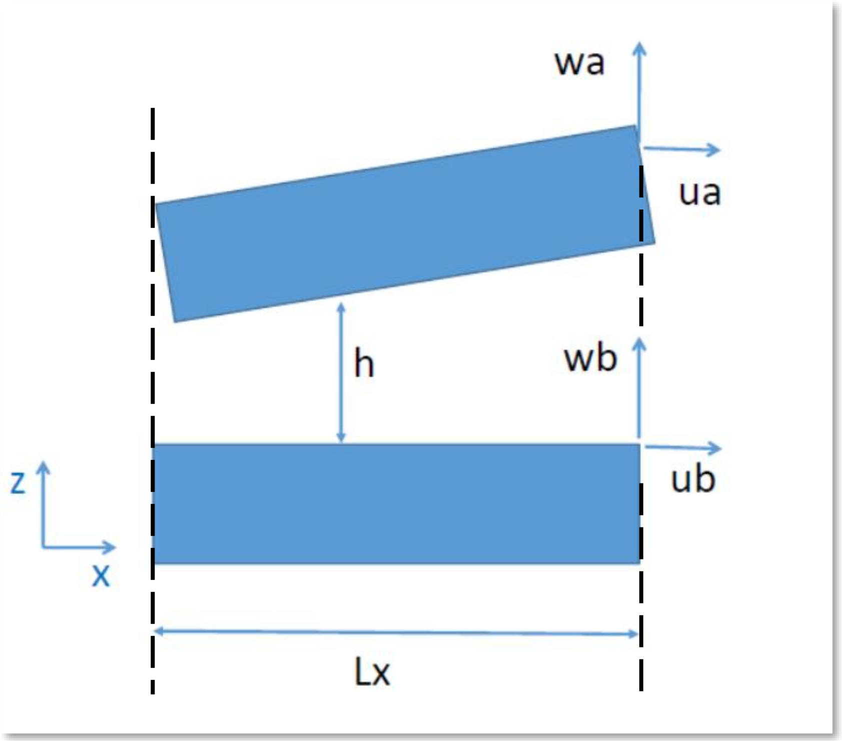
$\dfrac{\partial p}{\partial x} > 0$





Boundary conditions for pressure:

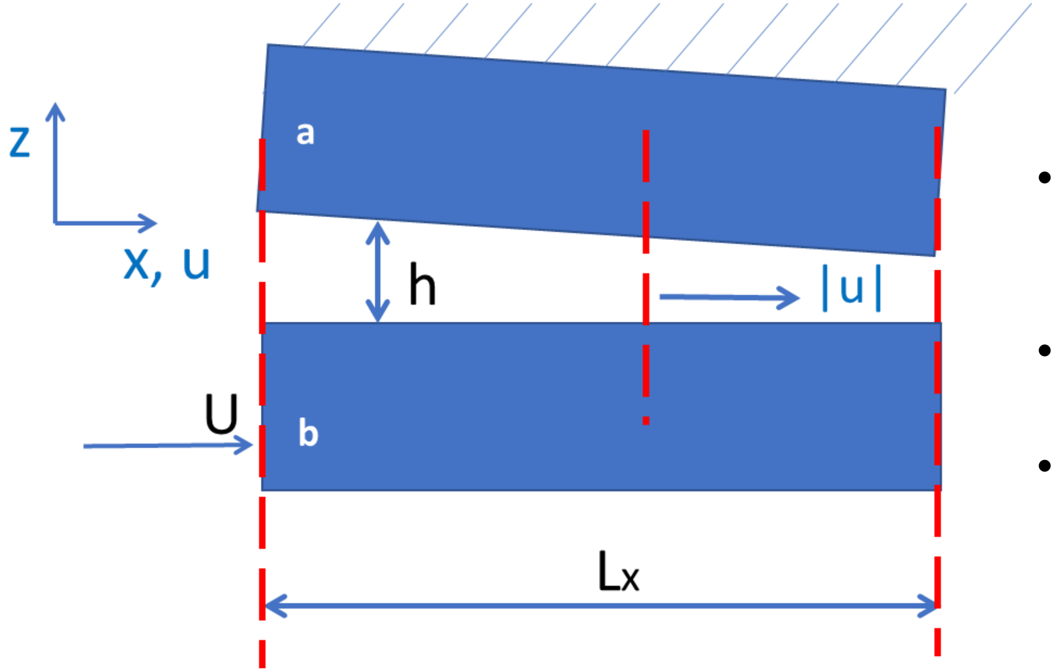
$p = p_a$ in the input and output sections



1D REYNOLDS' EQUATION

EXERCISE 1

Find the sign of the pressure derivative at the inlet and outlet section, and at an intermediate section of the thin film and plot the velocity distributions.



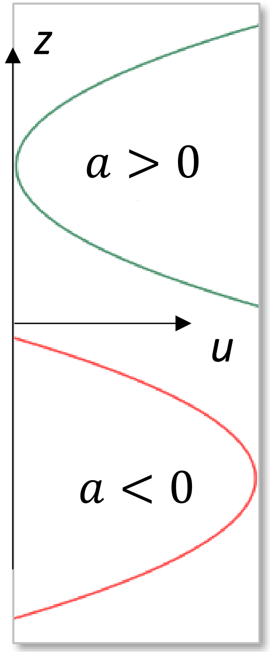
BCs:

$$u_a = 0 \qquad w_a = 0$$
$$u_b = +U \qquad w_b = 0$$

- Suppose to measure ambient pressure at the inlet and outlet sections: $p(0) = p_a$, $p(L_x) = p_a$
- Plot also the pressure profile.
- Remember that volume flow is the integral of velocity:

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x} z^2 - \frac{1}{2\mu}\frac{\partial p}{\partial x} zh + \frac{z}{h}(u_a - u_b) + u_b$$

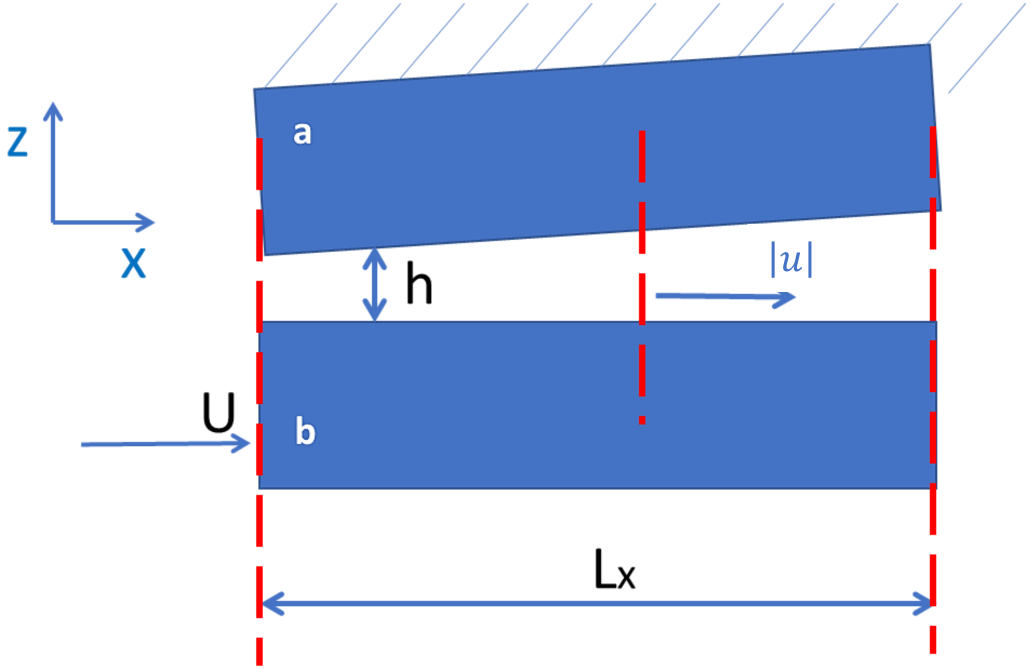
$$u = az^2 - bz + c$$



HINT: the volume flow is constant!

EXERCISE 2

Find the sign of the pressure derivative at the inlet and outlet section, and at an intermediate section of the thin film and plot the velocity distributions.



BCs:

$$u_a = 0 \qquad w_a = 0$$
$$u_b = +U \qquad w_b = 0$$

- Suppose to measure ambient pressure at the inlet and outlet sections: $p(0) = p_a$ $p(L_x) = p_a$
- Plot also the pressure profile.
- Remember that volume flow is the integral of velocity:

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}z^2 - \frac{1}{2\mu}\frac{\partial p}{\partial x}zh + \frac{z}{h}(u_a - u_b) + u_b$$

HINT: the volume flow is constant!

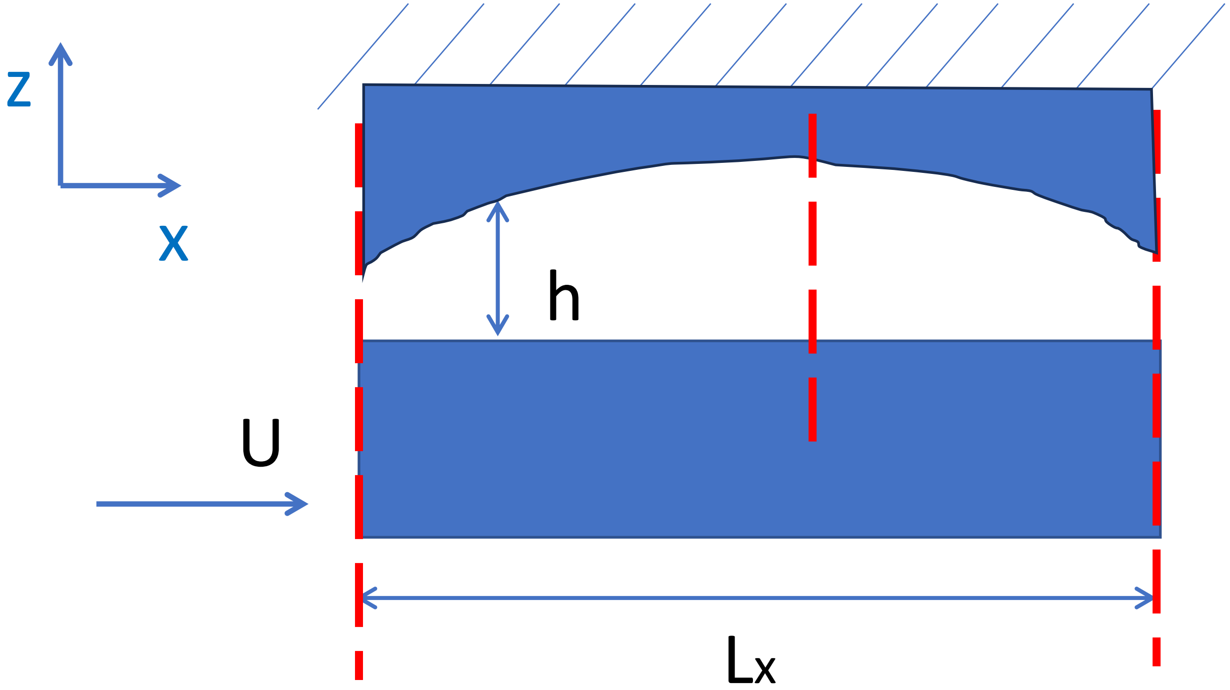
1D REYNOLDS' EQUATION

EXERCISE 3

Find the sign of the pressure derivative at the inlet and outlet section, and at an intermediate section of the thin film and plot the velocity distributions.

BCs:

$$u_\mathrm{a} = 0 \qquad w_\mathrm{a} = 0$$
$$u_\mathrm{b} = +U \qquad w_\mathrm{b} = 0$$



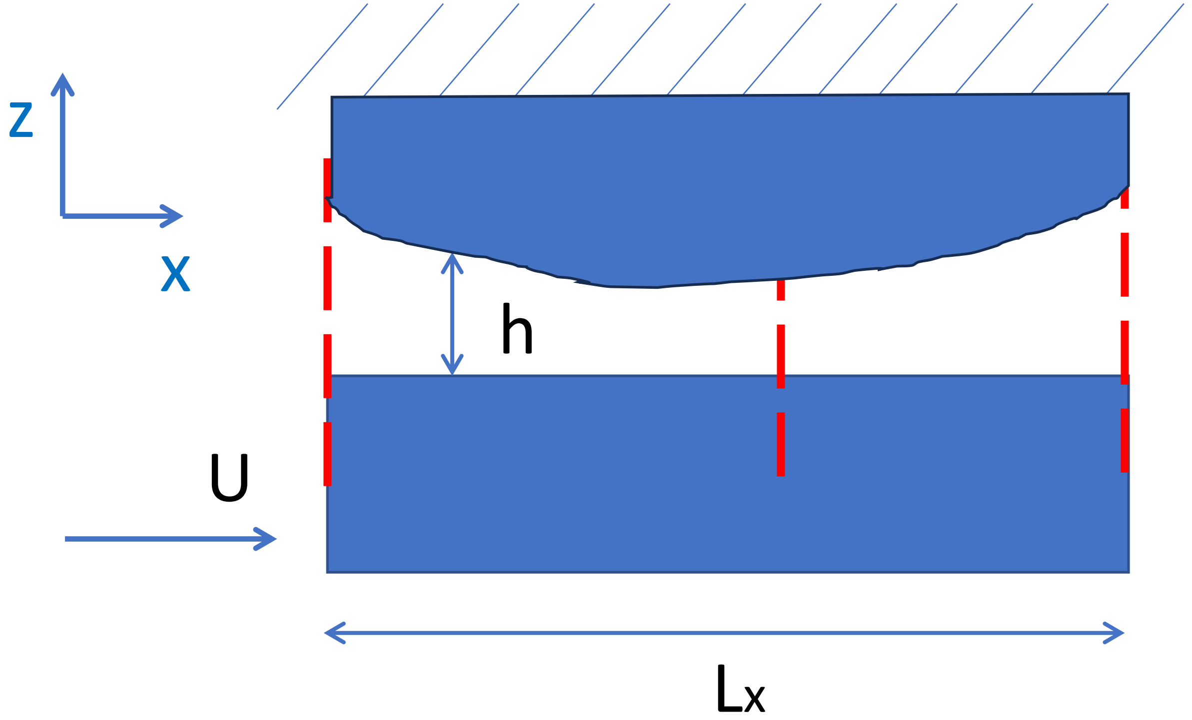
- Suppose to measure ambient pressure at the inlet and outlet sections: $p(0) = p_a$ $p(L_x) = p_a$
- Plot also the pressure profile.
- Remember that volume flow is the integral of velocity:

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}z^2 - \frac{1}{2\mu}\frac{\partial p}{\partial x}zh + \frac{z}{h}(u_a - u_b) + u_b$$

HINT: the volume flow is constant!

EXERCISE 4

Find the sign of the pressure derivative at the inlet and outlet section, and at an intermediate section of the thin film and plot the velocity distributions.



BCs:

$$u_\mathrm{a} = 0 \qquad w_\mathrm{a} = 0$$
$$u_\mathrm{b} = +U \qquad w_\mathrm{b} = 0$$

- Suppose to measure ambient pressure at the inlet and outlet sections: $p(0) = p_a$, $p(L_x) = p_a$
- Plot also the pressure profile.
- Remember that volume flow is <u>the integral of velocity</u>:

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}z^2 - \frac{1}{2\mu}\frac{\partial p}{\partial x}zh + \frac{z}{h}(u_a - u_b) + u_b$$

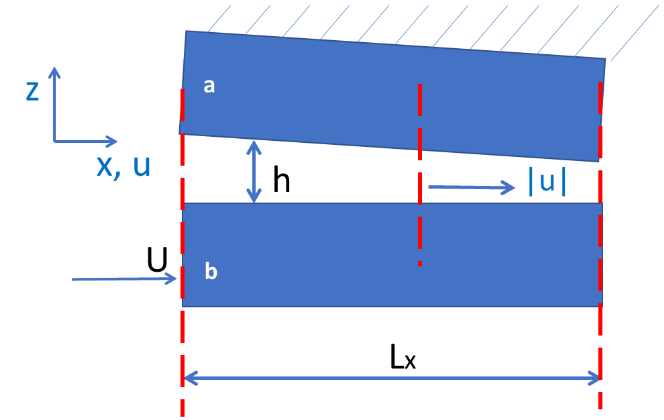
HINT: the volume flow is constant!

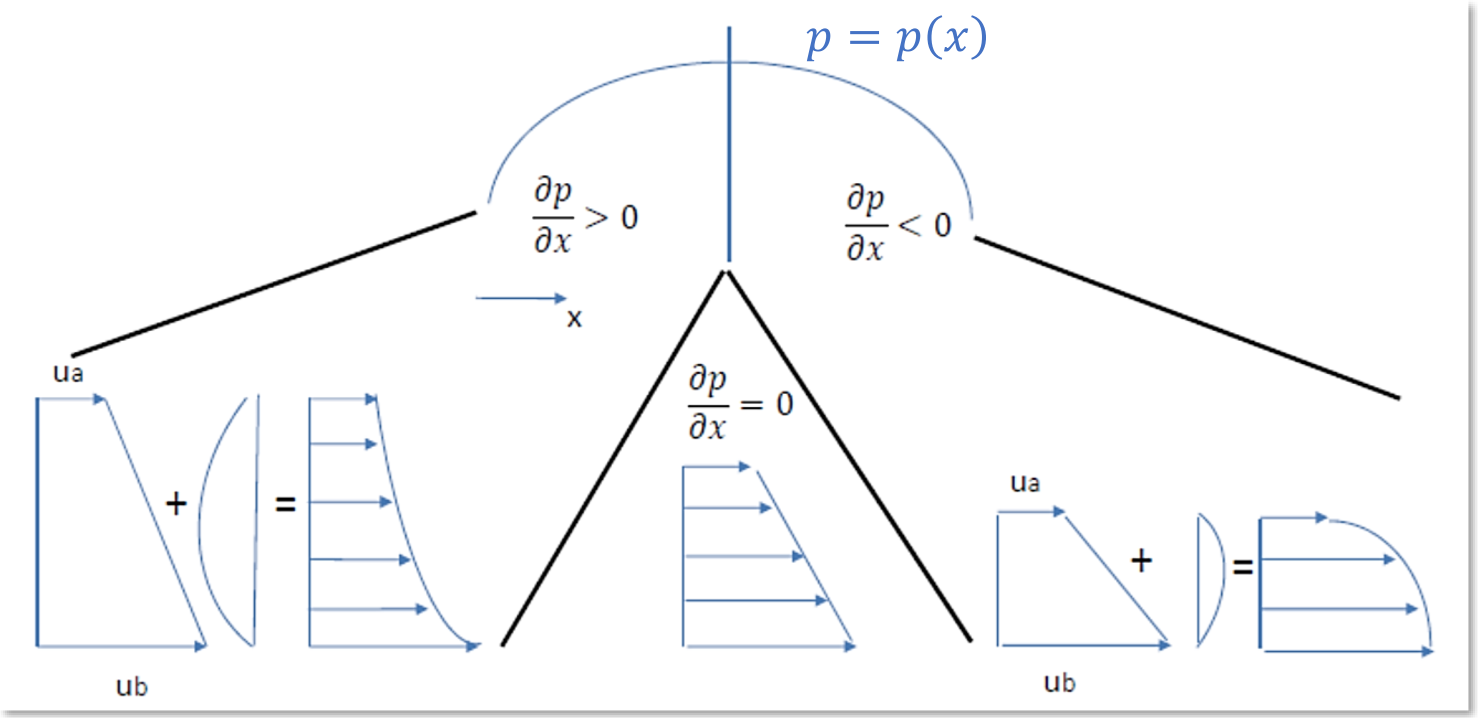
1D REYNOLDS' EQUATION

VOLUME FLOW

The volume flow (per unit of length along the y-direction) is calculated integrating speed *u* along the z-direction:

$$q_x = \int_0^h u\, dz = \underbrace{-\frac{h^3}{12\mu}\frac{\partial p}{\partial x}}_{\substack{\textit{Poiseuille flow} \\ \text{(pressure flow)}}} + \underbrace{\frac{u_a + u_b}{2}h}_{\substack{\textit{Couette flow} \\ \text{(drag flow)}}}$$



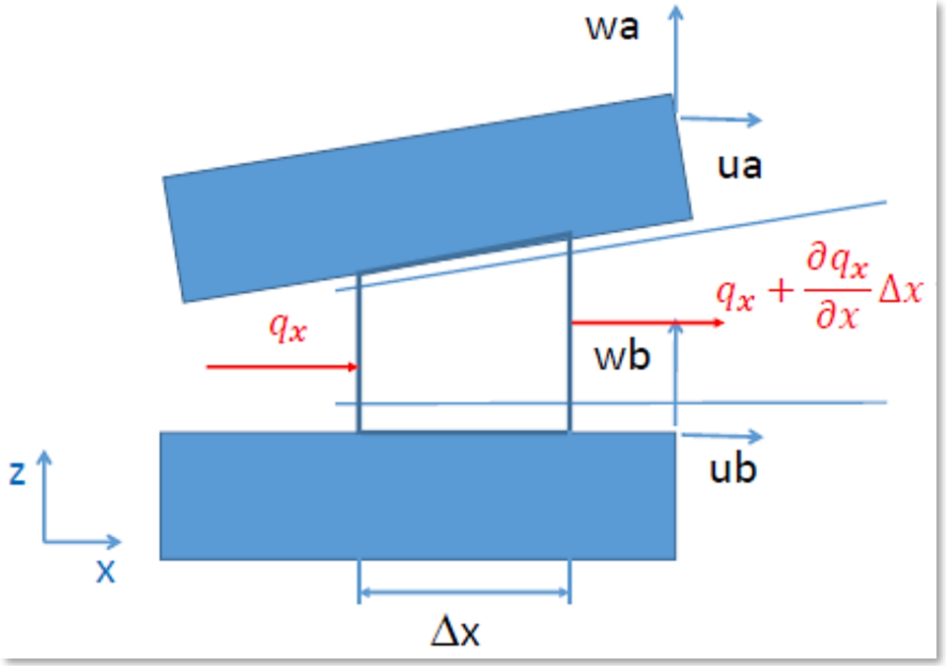


Convergent gap

1D REYNOLDS' EQUATION

MASS BALANCE

Let's consider an element of size $h\Delta x\Delta y$:



Input mass flow: $\rho q_x \Delta y$

Output mass flow: $\left(\rho q_x + \frac{\partial(\rho q_x)}{\partial x}\Delta x\right)\Delta y$

Inner mass variation: $\dot{m} = \frac{\partial(\rho h)}{\partial t}\Delta x\Delta y$

Continuity equation: $\rho q_x \Delta y - \left(\rho q_x + \frac{\partial(\rho q_x)}{\partial x}\Delta x\right)\Delta y = \frac{\partial(\rho h)}{\partial t}$

$$\frac{\partial(\rho h)}{\partial t} = \rho\frac{\partial h}{\partial t} + h\frac{\partial \rho}{\partial t}$$

$$-\frac{\partial(\rho q_x)}{\partial x} = \underbrace{\rho\frac{\partial h}{\partial t}}_{\text{Gap variations}} + \underbrace{h\frac{\partial \rho}{\partial t}}_{\text{Compressibility}}$$

1D REYNOLDS' EQUATION

MASS BALANCE – EFFECT OF GAP VARIATION

As the surfaces have speeds also along z direction, the derivative of the film thickness depends on:

- the components w_a and w_b
- the inclination $\frac{\partial h}{\partial x}$ of the upper surface

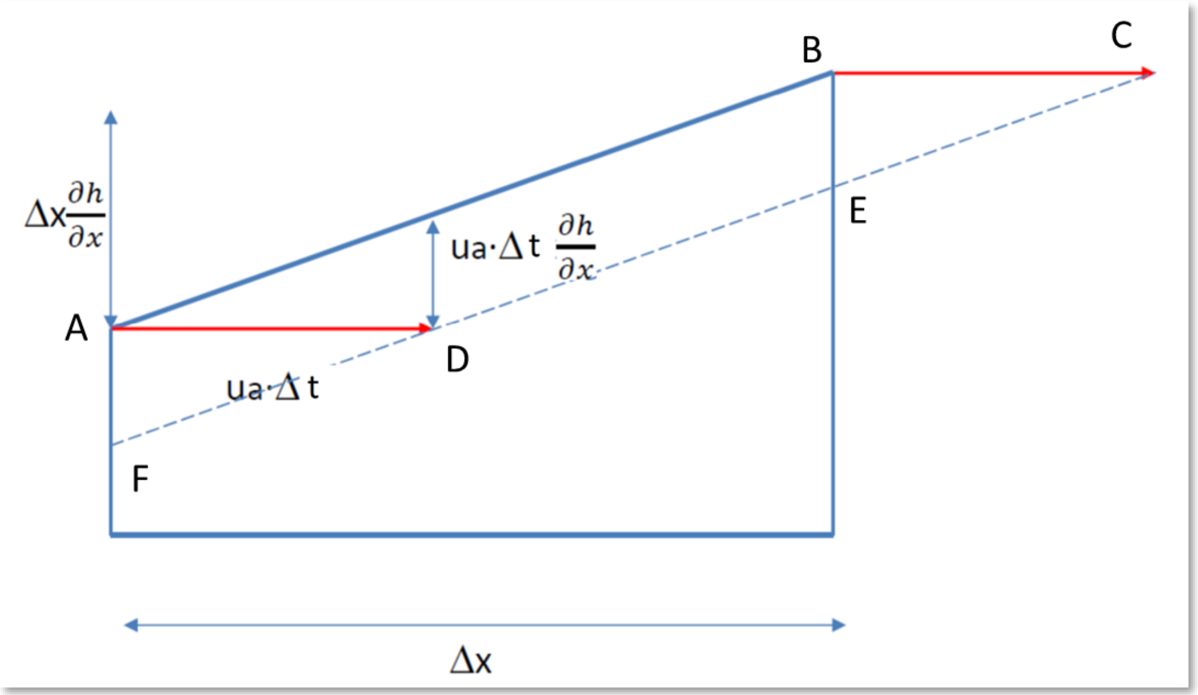
Due to the **speed w along the z-direction** of the walls, the volume increases of quantity

$$\Delta V = (w_a - w_b)\Delta t \Delta x \Delta y$$

Due to the **translation along the x-direction** of the upper surface, the upper volume ABCD here below indicated is subtracted:

$$\Delta V = Area(ABCD)\Delta y =$$

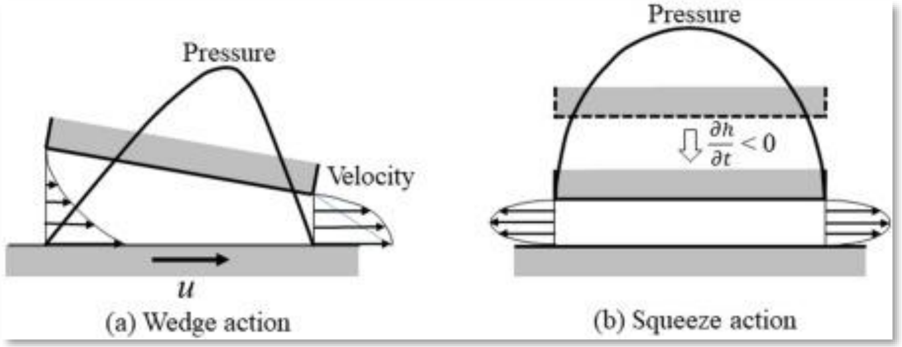
$$= Area(ABEF)\Delta y = \quad u_a \Delta t \frac{\partial h}{\partial x} \Delta x \Delta y$$



1D REYNOLDS' EQUATION

DIFFERENTIAL (OR LOCAL) FORM

$$\frac{\Delta V}{\Delta t\Delta x\Delta y} \rightarrow \frac{\partial h}{\partial t} = \underbrace{(w_a - w_b)}_{\text{squeeze effect}} - \underbrace{u_a\frac{\partial h}{\partial x}}_{\text{wedge effect}}$$



(a) Wedge action

(b) Squeeze action

Continuity equation: $$-\frac{\partial(\rho q_x)}{\partial x} = \rho\left(w_a - w_b - u_a\frac{\partial h}{\partial x}\right) + h\frac{\partial \rho}{\partial t}$$

Volume flow per unit length $$q_x = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x} + \frac{u_a + u_b}{2}h$$

1D Reynolds equation (1D RE)

$$\boxed{\underbrace{\frac{\partial}{\partial x}\left(\frac{\rho h^3}{12\mu}\frac{\partial p}{\partial x}\right)}_{\substack{\text{Poiseuille flow}\\\text{(pressure flow)}}} - \underbrace{\frac{\partial}{\partial x}\left(\rho h\frac{u_a + u_b}{2}\right)}_{\substack{\text{Couette flow}\\\text{(drag flow)}}} - \underbrace{\rho\left(w_a - w_b - u_a\frac{\partial h}{\partial x}\right)}_{\text{Squeeze flow}} - \underbrace{h\frac{\partial \rho}{\partial t}}_{\text{Compressibility}} = 0}$$

2D REYNOLDS' EQUATION

DIFFERENTIAL (OR LOCAL) FORM

Boundary conditions for the speed:

$$\begin{cases} u(z=h) = u_\mathrm{a} \\ u(z=0) = u_\mathrm{b} \end{cases} \qquad \begin{cases} v(z=h) = v_a \\ v(z=0) = v_\mathrm{b} \end{cases} \qquad \begin{cases} w(z=h) = w_\mathrm{a} \\ w(z=0) = w_\mathrm{b} \end{cases}$$

2D Thin film

$$\frac{h_0}{l_0} \ll 1, \quad \rightarrow \quad \left(\frac{\partial w}{\partial x}, \frac{\partial u}{\partial x}\right) \ll \frac{\partial u}{\partial z} \qquad \left(\frac{\partial w}{\partial y}, \frac{\partial v}{\partial y}\right) \ll \frac{\partial v}{\partial z}$$

Viscous, Newtonian fluid

$$\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \cong \mu\,\frac{\partial u}{\partial z} \qquad \tau_{zy} = \mu\left(\frac{\partial \mathrm{v}}{\partial z} + \frac{\partial w}{\partial y}\right) \cong \mu\,\frac{\partial \mathrm{v}}{\partial z}$$

2D volume flow

$$q_x = \int_0^h u\,dz = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x} + \frac{u_a + u_b}{2}h \qquad q_y = \int_0^h v\,dz = -\frac{h^3}{12\mu}\frac{\partial p}{\partial y} + \frac{v_a + v_b}{2}h$$

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\rho h^3}{12\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\rho h^3}{12\mu}\frac{\partial p}{\partial y}\right) - \frac{\partial}{\partial x}\left(\rho h\frac{u_a + u_b}{2}\right) - \frac{\partial}{\partial y}\left(\rho h\frac{v_a + v_b}{2}\right) - \rho\left(w_a - w_b - u_a\frac{\partial h}{\partial x} - v_a\frac{\partial h}{\partial y}\right) - h\frac{\partial \rho}{\partial t} = 0}$$

2D Reynolds equation (2D RE)

SOLUTION OF SIMPLE CASES

Analytical & numerical solution of simple cases

- 1D hydrodynamic slider
- 1D slider with normal squeeze motion
- 1D slider, hydrostatic effect

Numerical solution of simple cases

- Lubricated cylinder on plane
- Journal bearing with infinite length
- 2D hydrodynamic slider

SOLUTION OF SIMPLE CASES

1D HYDRODYNAMIC SLIDER

Incompressible fluid (ρ is constant)

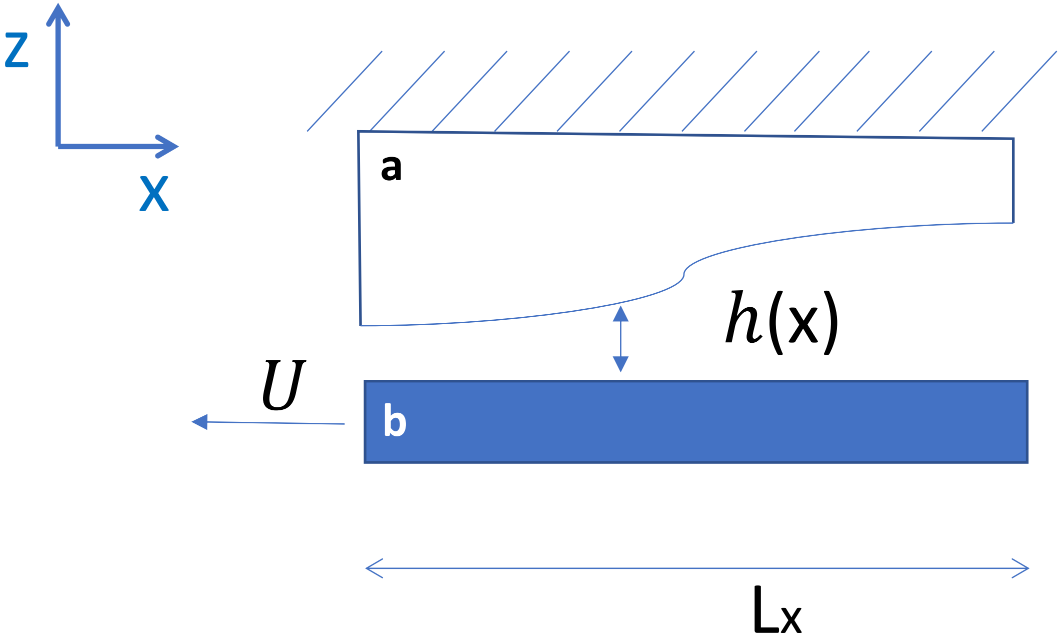
$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)+6\mu U\frac{\partial h}{\partial x}=0$$

First integration step:
$$\frac{\partial p}{\partial x}=-\frac{6\mu U}{h^2}+\frac{c}{h^3}$$

Integrating twice:
$$p(x)=p_a-6\mu U\int_0^x\frac{dx}{h^2}+c\int_0^x\frac{dx}{h^3}$$

where the constant c is calculated when $p(L_x)=p_a$ is known:
$$c\;=\;6\mu U\frac{\int_0^{L_x}\frac{1}{h^2}dx}{\int_0^{L_x}\frac{1}{h^3}dx}\;=\;6\mu U h_0$$

Generic profile *h(x)*



BCs:

$u_a=0$ $\quad p(0)=p_a$

$u_b=-U$ $\quad p(L_x)=p_a$

SOLUTION OF SIMPLE CASES

1D HYDRODYNAMIC SLIDER

$$p(x) = p_a - 6\mu U \int_0^x \frac{dx}{h^2} + 6\mu U h_0 \int_0^x \frac{dx}{h^3}$$

$$\frac{\partial p}{\partial x} = -\frac{6\mu U}{h^2}\left(1 - \frac{h_0}{h}\right) = \frac{6\mu U}{h^2}\left(\frac{h_0 - h}{h}\right)$$

From this expression, it is evident that when

$h > h_0$ $\quad \frac{\partial p}{\partial x} < 0$

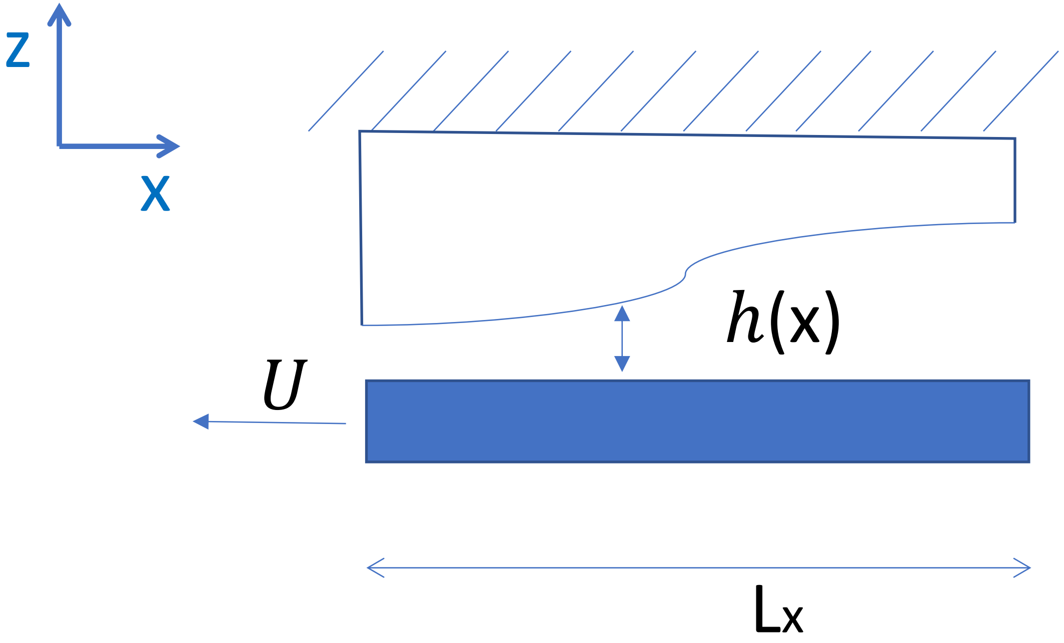
$h < h_0$ $\quad \frac{\partial p}{\partial x} > 0$

$h = h_0$ $\quad \frac{\partial p}{\partial x} = 0$

For the mass conservation, the lubricant flow q_x is constant along the *x direction*:

$$q_x = -\frac{U}{2} h_0$$

Generic profile *h(x)*



To have pressure generation it is needed:

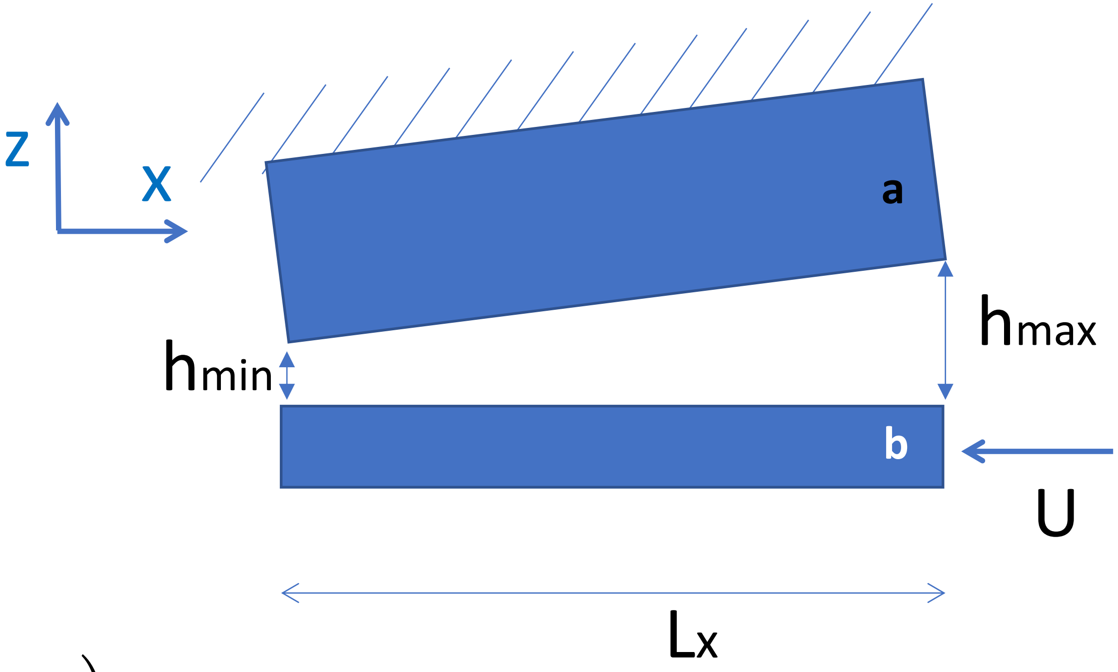
- Viscosity $\mu \neq 0$
- Speed $U \neq 0$
- Gap gradient $\frac{dh}{dx} \neq 0$

SOLUTION OF SIMPLE CASES

LINEAR PROFILE SLIDER

$$h = h_{min}\left(1 + \frac{h_{max} - h_{min}}{L_x}\frac{x}{h_{min}}\right) = h_{min}\left(1 + m\frac{x}{L_x}\right)$$

$$m = \frac{h_{max} - h_{min}}{h_{min}} \qquad slope = \frac{h_{max} - h_{min}}{L_x} = \frac{m \cdot h_{min}}{L_x}$$



Let us calculate the expression of h0:

$$\int_0^{L_x} \frac{1}{h^2}dx = \frac{L_x}{h_{min}^2} \cdot \frac{1}{1+m}, \qquad \int_0^{L_x} \frac{1}{h^3}dx = \frac{-L_x/m}{2h_{min}^3}\left(\frac{1}{(1+m)^2} - 1\right)$$

BCs:

$$u_a = 0 \qquad p(0) = p_a$$
$$u_b = -U \qquad p(L_x) = p_a$$

$$h_0 = \frac{\int_0^{L_x}\frac{1}{h^2}dx}{\int_0^{L_x}\frac{1}{h^3}dx} = \frac{2h_{min}}{2+m}(1+m)$$

Film thickness at which we have the maximum of pressure

SOLUTION OF SIMPLE CASES

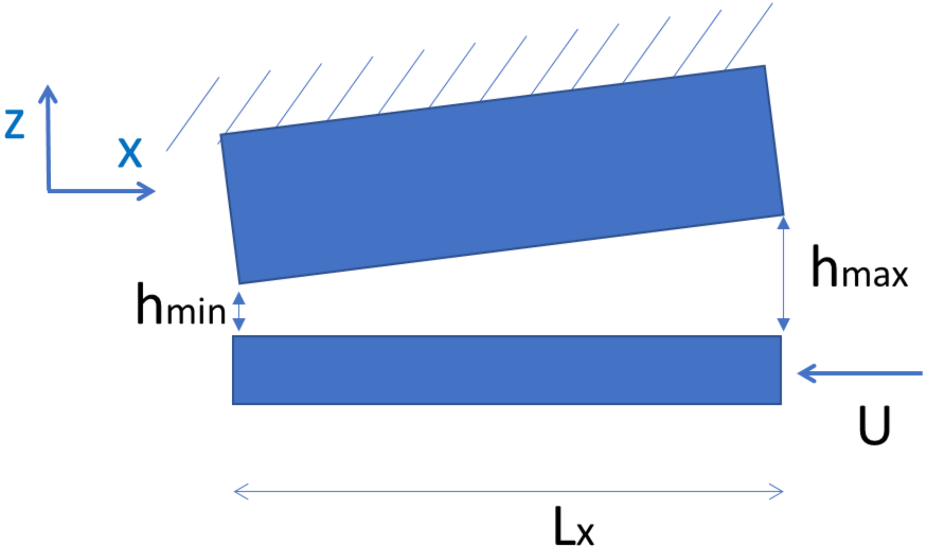
LINEAR PROFILE SLIDER - II

$$\frac{\partial p}{\partial x} = -\frac{6\mu U}{h^2}\left(1-\frac{h_0}{h}\right) = \frac{6\mu U}{h^2}\left(\frac{h_0-h}{h}\right)$$

Integrating once:

$$p - p_a = \frac{L_x/m}{h_{min}^2}6\mu U\left[\frac{-m\frac{x}{L_x}}{1+m\frac{x}{L_x}} - \frac{(1+m)}{2+m}\left(\frac{1}{\left(1+m\frac{x}{L_x}\right)^2}-1\right)\right]$$

$$p - p_a = -\frac{6\mu U}{h_{min}^2}\frac{x}{1+m\frac{x}{L_x}}\left[1-\frac{\frac{(1+m)}{2+m}\left(2+m\frac{x}{L_x}\right)}{\left(1+m\frac{x}{L_x}\right)}\right]$$



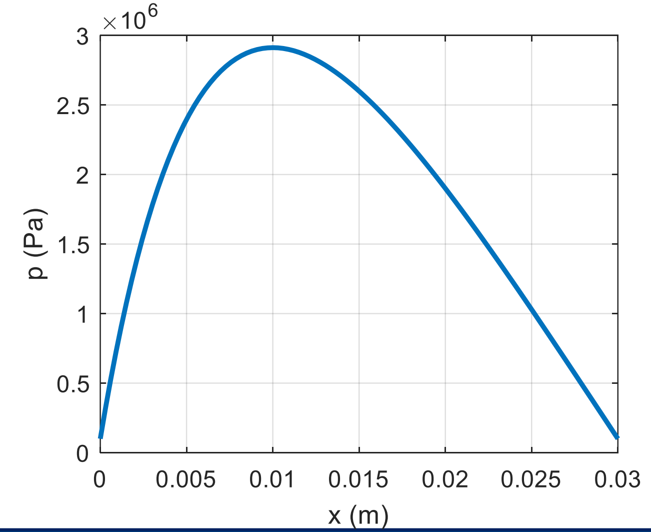
h_{min} = 20 μm

h_{max} = 40 μm

Lx = 30 mm

U = 1 m/s

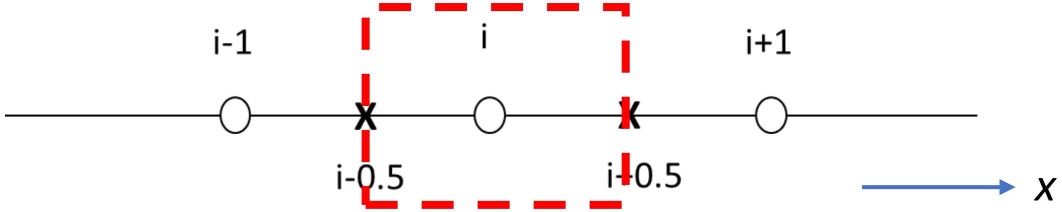
μ = 0.15 Pa·s



SOLUTION OF SIMPLE CASES

1D HYDRODYNAMIC SLIDER, NUMERICAL SOLUTION

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}+\frac{Uh}{2}\right)=0 \quad \Rightarrow \quad \frac{h^3}{12\mu}\frac{\partial p}{\partial x}+\frac{Uh}{2}=cost$$



N+2 nodes along x direction, from i = 0 to i = N+1

Discretization of the 1D Reynolds equation with **finite difference method (FDM):**

$$\left.\left(\frac{1}{12\mu}\frac{dp}{dx}h^3+\frac{Uh}{2}\right)\right|_{i+1/2}=\left.\left(\frac{1}{12\mu}\frac{dp}{dx}h^3+\frac{Uh}{2}\right)\right|_{i-1/2}$$

The flow is evaluated at the boundaries of the control volume

$$\frac{1}{12\mu}\cdot\left.\frac{\Delta p}{\Delta x}\right|_{i+\frac{1}{2}}h^3_{i+\frac{1}{2}}+\frac{U}{2}h_{i+\frac{1}{2}}-\frac{1}{12\mu}\left.\frac{\Delta p}{\Delta x}\right|_{i-\frac{1}{2}}h^3_{i-\frac{1}{2}}+\frac{U}{2}h_{i-\frac{1}{2}}=0$$

NB: p gradient is always estimated at mid-points!

$$\frac{1}{12\mu}\frac{p_{i+1}-p_i}{\Delta x}h^3_{i+\frac{1}{2}}+\frac{U}{2}h_{i+\frac{1}{2}}-\frac{1}{12\mu}\frac{p_i-p_{i-1}}{\Delta x}h^3_{i-\frac{1}{2}}+\frac{U}{2}h_{i-\frac{1}{2}}=0$$

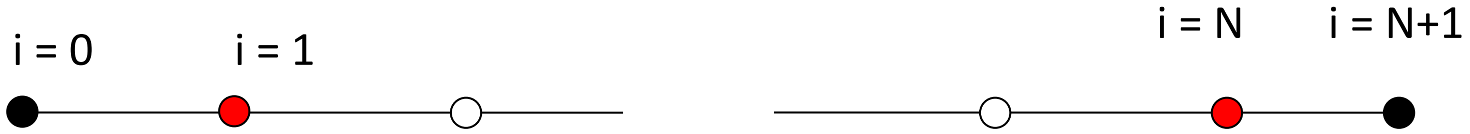
SOLUTION OF SIMPLE CASES

1D HYDRODYNAMIC SLIDER, NUMERICAL SOLUTION

BCs:

$$p_{i=0} = p_a$$

$$p_{i=N+1} = p_a$$



There results in *N* linear algebraic equations in *N* variables:

$$\underbrace{p_{i-1}\frac{h^3_{i-\frac{1}{2}}}{\Delta x}}_{\in b_1 \quad if \quad i=1} - p_i \frac{h^3_{i-\frac{1}{2}} + h^3_{i+\frac{1}{2}}}{\Delta x} + \underbrace{p_{i+1}\frac{h^3_{i+\frac{1}{2}}}{\Delta x}}_{\in b_N \quad if \quad i=N} = -12\mu\frac{U}{2}\left(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}\right)$$

NB:

In matrix form: $\boldsymbol{Ap} = \boldsymbol{b}$

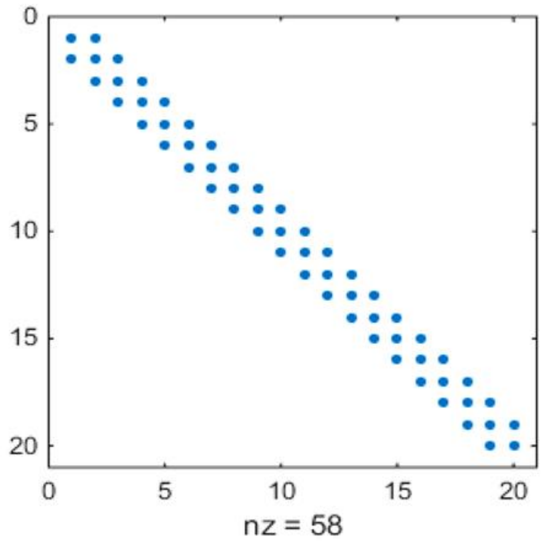
$$A_{i,i-1}p_{i-1} + A_{i,i}p_i + A_{i,i+1}p_{i+1} = b_i$$

$$A_{i,i-1} = \frac{1}{2}\frac{{h_{i-1}}^3 + {h_i}^3}{\Delta x}$$

$$A_{i,i+1} = \frac{1}{2}\frac{{h_i}^3 + {h_{i+1}}^3}{\Delta x}$$

$$A_{i,i} = -\frac{1}{2}\frac{\left({h_{i-1}}^3 + 2{h_i}^3 + {h_{i+1}}^3\right)}{\Delta x}$$

$$b_i = -\frac{12\mu U}{4}\left(h_{i+1} - h_{i-1}\right)$$

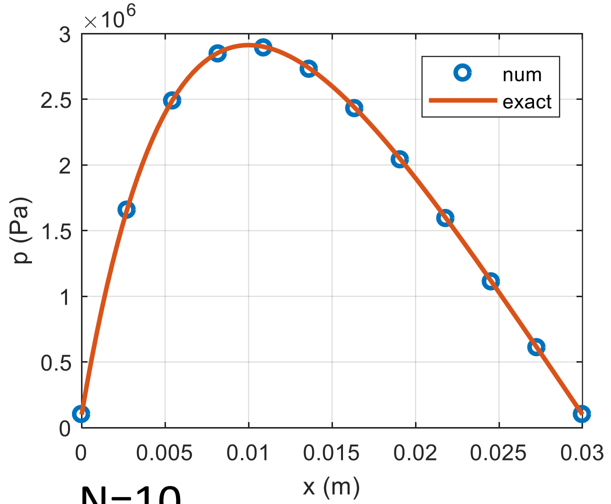


SOLUTION OF SIMPLE CASES

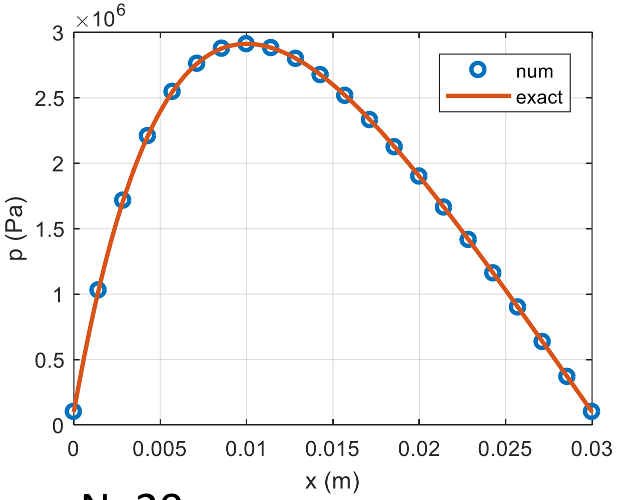
1D HYDRODYNAMIC SLIDER, NUMERICAL SOLUTION

$$\frac{1}{12\mu\Delta x}\begin{bmatrix} -\left(h^3_{\frac{1}{2}}+h^3_{\frac{3}{2}}\right) & h^3_{\frac{3}{2}} & 0\,\dots & & \\ \ddots & \ddots & \ddots \quad \ddots & \ddots & \\ 0 & h^3_{i-\frac{1}{2}} & -\left(h^3_{i-\frac{1}{2}}+h^3_{i+\frac{1}{2}}\right) & h^3_{i+\frac{1}{2}} & 0 \\ & & \ddots & \ddots & \ddots \\ & & 0 & h^3_{N-\frac{1}{2}} & -\left(h^3_{N-\frac{1}{2}}+h^3_{N+\frac{1}{2}}\right) \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_i \\ \vdots \\ p_N \end{bmatrix} = \frac{-U}{2}\begin{bmatrix} h_{\frac{3}{2}}-h_{\frac{1}{2}} \\ \vdots \\ h_{i+\frac{1}{2}}-h_{i-\frac{1}{2}} \\ \vdots \\ h_{N+\frac{1}{2}}-h_{N-\frac{1}{2}} \end{bmatrix} - \frac{p_a}{12\mu\Delta x}\begin{bmatrix} h^3_{\frac{1}{2}} \\ 0 \\ \vdots \\ 0 \\ h^3_{N+\frac{1}{2}} \end{bmatrix}$$

Direct solving method: $\boldsymbol{p} = \boldsymbol{A}\backslash\boldsymbol{b}$



N=10



N=20

SOLUTION OF SIMPLE CASES

ASSIGNMENT A1

IMPLEMENT A MATLAB SCRIPT THAT SOLVES THE 1D HYDRODYNAMIC SLIDER SYSTEM AND OBTAINS THE PRESSURE DISTRIBUTION INTO THE GAP.

COMPARE IT WITH THE ANALYTIC SOLUTION

```
%% Clear workspace and
variables
clear all
close all
clc
```

```
%% Preprocessing
% Introducing variables
pa = 1e5;        %Pa
mu = 150e-3;     %Pa*s viscosity
hmin = 20e-6;    %m
hmax = 40e-6;    %m
L = 30e-3;       %m
U = 1;           %m/s
N = ...;          %n. of internal nodes
```

```
% Discretization
x = ...;  %nodes position
deltax = ...;
gap = ...;
h = ...; % internal nodes

...
```

SOLUTION OF SIMPLE CASES

Analytical & numerical solution of simple cases

- 1D hydrodynamic slider
- 1D slider with normal squeeze motion
- 1D slider, hydrostatic effect

Numerical solution of simple cases

- Lubricated cylinder on plane
- Journal bearing with infinite length
- 2D hydrodynamic slider

SOLUTION OF SIMPLE CASES

1D SLIDER, NORMAL SQUEEZE MOTION

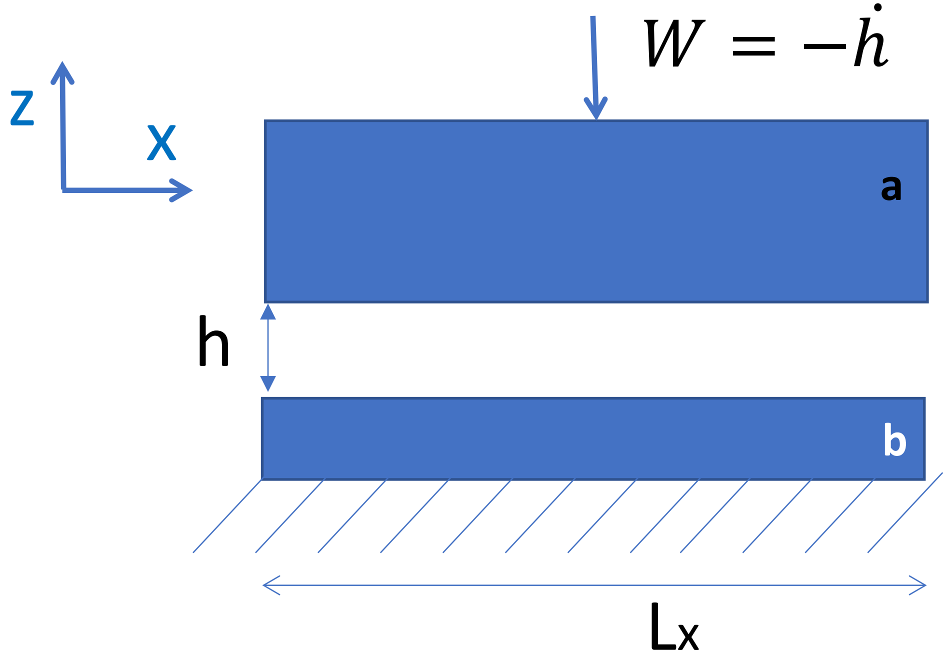
Incompressible fluid (ρ is constant)

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{12\mu}\frac{\partial p}{\partial x}\right)-\cancel{\frac{\partial}{\partial x}\left(\rho h\frac{u_a+u_b}{2}\right)}-\rho\frac{\partial h}{\partial t}-\cancel{h\frac{\partial \rho}{\partial t}}=0$$

$$\Downarrow$$

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)=12\mu\dot{h}$$

NB: *h constant along the x-direction!*
No tilting of the pad is considered



$$\frac{\partial^2 p}{\partial x^2}=-\frac{12\mu W}{h^3}$$

$$\rightarrow \quad \frac{dp}{dx}=-\frac{12\mu W}{h^3}x+c_1$$

$$\rightarrow \quad p=-\frac{6\mu W}{h^3}x^2+c_1x+c_2$$

BCs:

$u_{\mathrm{a}}=0$
$u_{\mathrm{b}}=0$
$p(0)=p_a$
$p(L_x)=p_a$

SOLUTION OF SIMPLE CASES

1D SLIDER, NORMAL SQUEEZE MOTION

Boundary conditions:

$$p(0) = c_2 = p_a \qquad p(L_x) = -\frac{6\mu W}{h^3} {L_x}^2 + c_1 L_x + p_a = p_a$$

$$\frac{dp}{dx} = \frac{6\mu W}{h^3}(L_x - 2x)$$

$$p = \frac{6\mu W}{h^3} x(L_x - x) + p_a$$

Volume flow:

$$q_x = \int_0^h u\, dz = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x} + \cancel{\frac{u_a + u_b}{2}} h$$

$$q_x = -\frac{W}{2}(L_x - 2x)$$

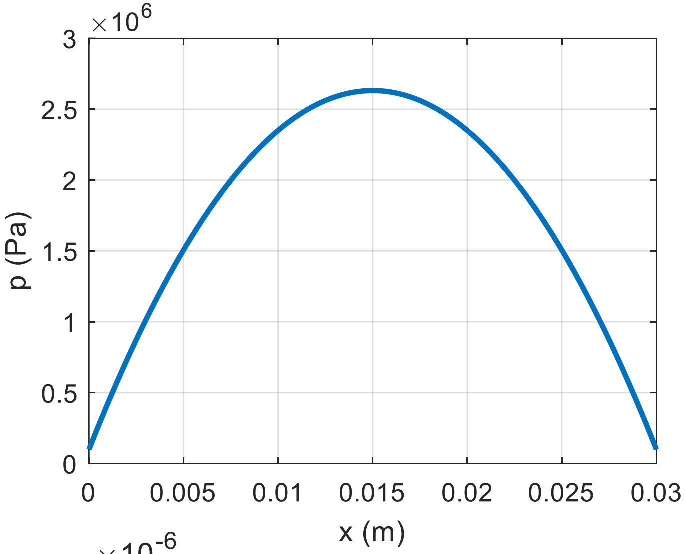
Example with:

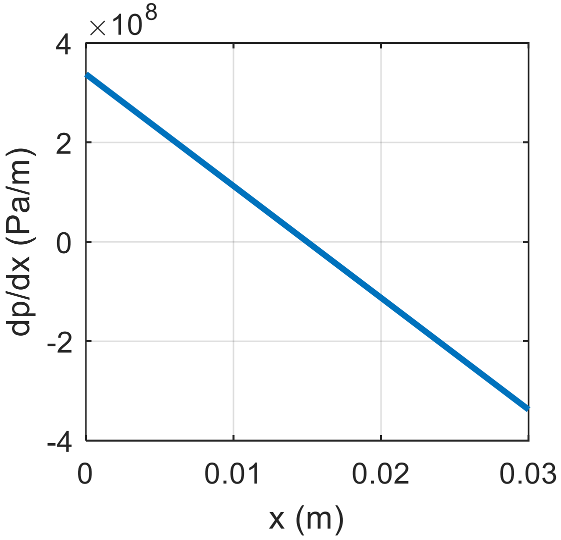
W=0.1mm/s

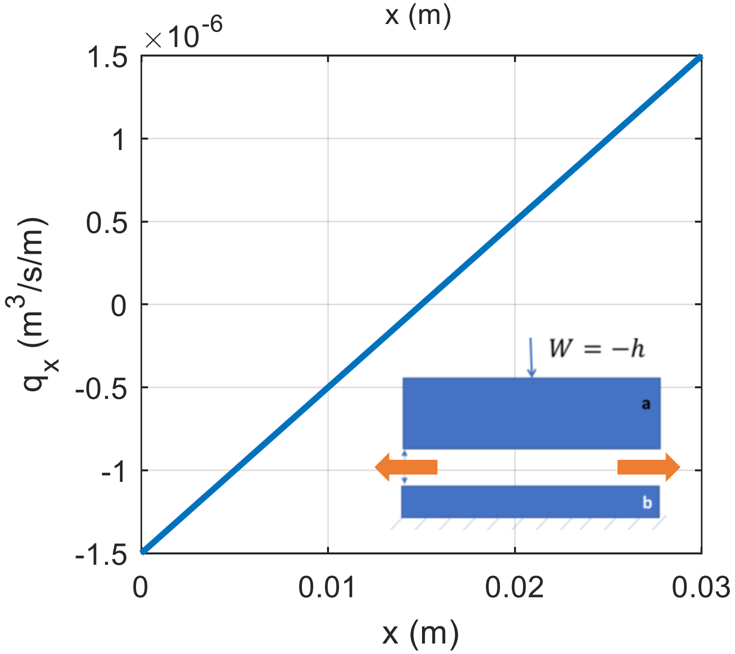
h=20mm

Lx=30mm

m=0.15Pa·s







SOLUTION OF SIMPLE CASES

ASSIGNMENT A2

DERIVE THE DISCRTE FORM OF THE REYNOLDS EQUATION AND IMPLEMENT A MATLAB SCRIPT THAT SOLVES THE 1D SLIDER WITH SQUEEZE MOTION AND OBTAINS THE PRESSURE DISTRIBUTION INTO THE GAP.

COMPARE WITH THE ANALYTIC SOLUTION

MATLAB

```
%% Clear workspace and variables
clear all
close all
clc
```

```
%% Preprocessing
% Introducing variables
pa = 1e5;         %Pa
mu = 150e-3;      %Pa*s viscosity
W = -0.1e-3;       %m
L = 30e-3;        %m
N = ...;           %n. of internal nodes
h = 80 e-6;       %m
```

```
% Discretization
x = ...;         %nodes position
deltax = ...;

...
```

SOLUTION OF SIMPLE CASES

Analytical & numerical solution of simple cases

- 1D hydrodynamic slider
- 1D slider with normal squeeze motion
- 1D slider, hydrostatic effect

Numerical solution of simple cases

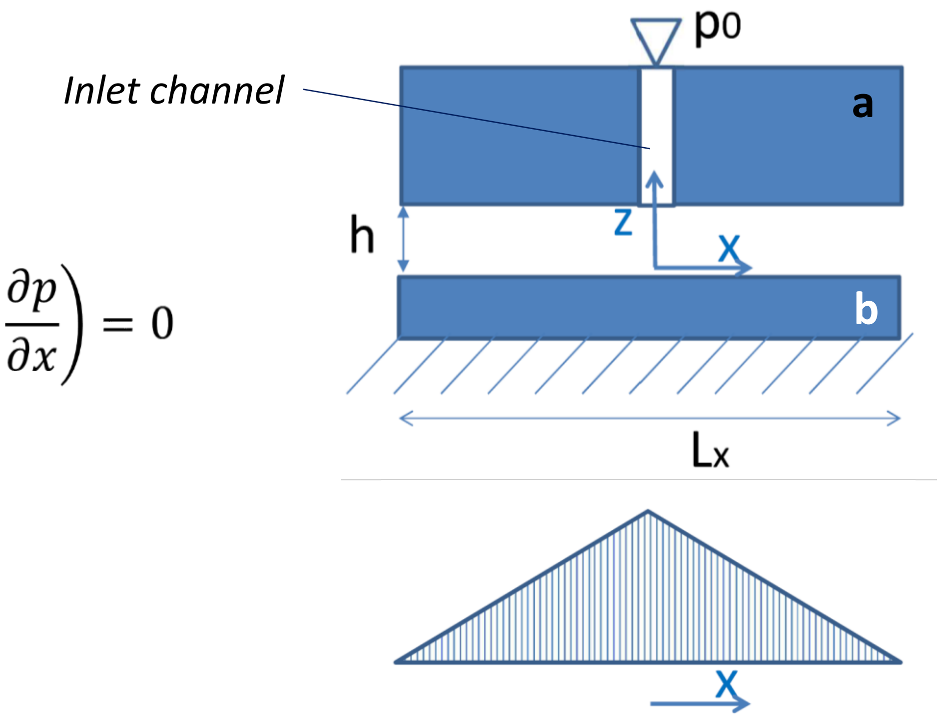
- Lubricated cylinder on plane
- Journal bearing with infinite length
- 2D hydrodynamic slider

SOLUTION OF SIMPLE CASES

1D SLIDER, HYDROSTATIC EFFECT

Incompressible fluid (ρ is constant)
Pressure drop into the inlet channel is neglected

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{12\mu}\frac{\partial p}{\partial x}\right)-\cancel{\frac{\partial}{\partial x}\left(\rho h\frac{u_a+u_b}{2}\right)}-\rho\cancel{\frac{\partial h}{\partial t}}-h\cancel{\frac{\partial \rho}{\partial t}}=0 \quad \Rightarrow \quad \frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)=0$$



To have pressure generation a pump is used that can provide:

- Constant flow
- Constant inlet pressure

$$p=p_a+(p_0-p_a)\left(1-\frac{2x}{L_x}\right)$$

$$q_x=\frac{h^3}{12\mu}\frac{2}{L_x}(p_0-p_a)$$

BCs:

$u_a = 0$
$u_b = 0$

$p(0) = p_a$
$p(L_x) = p_a$
$p(L_x/2) = p_0$

SOLUTION OF SIMPLE CASES

Analytical & numerical solution of simple cases

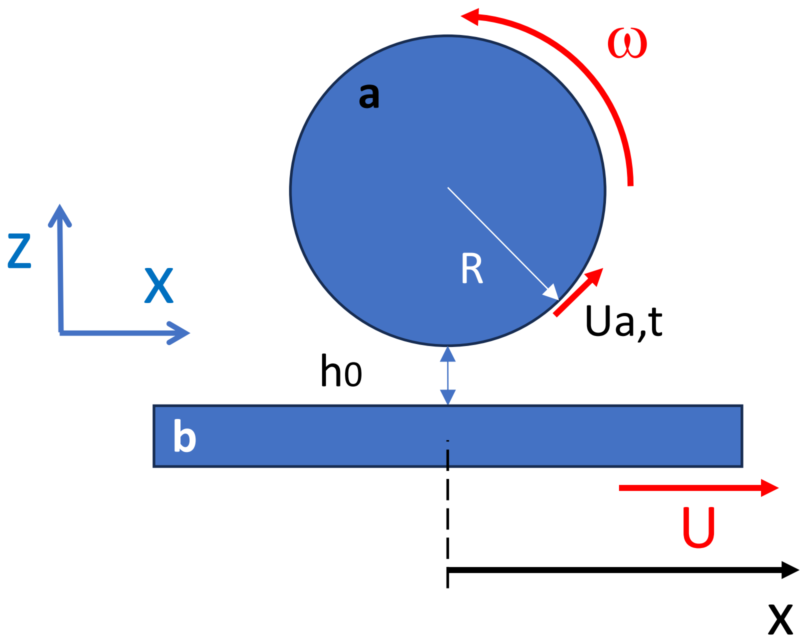
- 1D hydrodynamic slider
- 1D slider with normal squeeze motion
- 1D slider, hydrostatic effect

Numerical solution of simple cases

- Lubricated cylinder on plane
- Journal bearing with infinite length
- 2D hydrodynamic slider

SOLUTION OF SIMPLE CASES

LUBRICATED CYLINDER ON PLANE



Hypotheses:

- Both cylinder and plane are rigid → *rigid-hydrodynamic problem*
- The cylinder rotates but its centre is fixed in space
- The plane is moving rightwards with speed U
- Distance h0 is given
- Incompressible fluid → but *no cavitation*
- Infinite cylinder length

BCs:

$$u_{\mathrm{a,t}} = \omega R \qquad p(?) = p_a$$
$$u_{\mathrm{b}} = U$$

Which BCs to be used?

SOLUTION OF SIMPLE CASES

LUBRICATED CYLINDER ON PLANE

Types of boundary conditions:

- **Dirichlet:** *pressure is known*

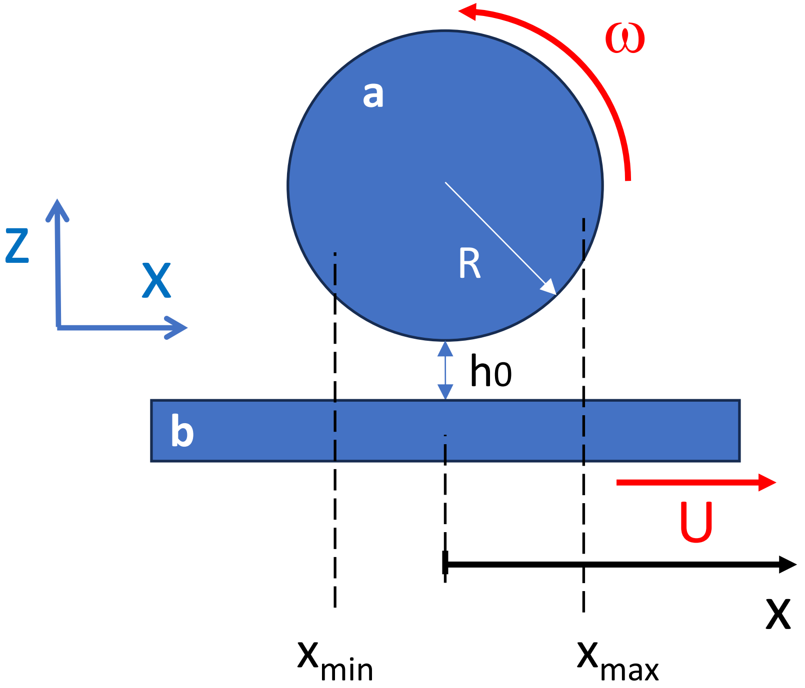
  $$p(x_{min}) = p_a; \quad p(x_{max}) = p_a$$

- **Neumann:** *pressure gradient is known*

  $$\frac{dp}{dx}(x_{max}) = 0$$

- **Robin:** *A combination of pressure and pressure gradient is known*

  $$ap(x_{max}) + b\frac{dp}{dx}(x_{max}) = 0$$



SOLUTION OF SIMPLE CASES

LUBRICATED CYLINDER ON PLANE

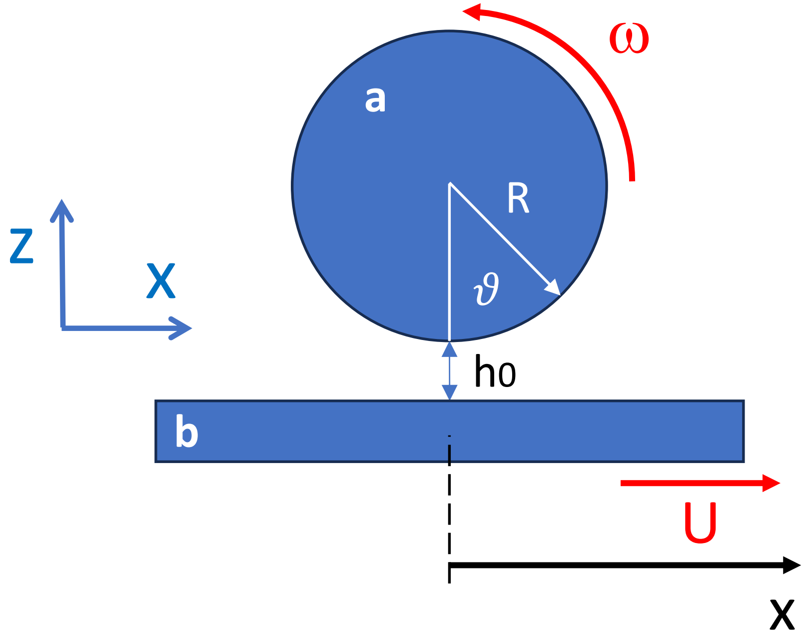
Film thickness: $h = h_0 + R(1 - \cos\vartheta)$

$$x = R\sin\vartheta$$

$$\cos\vartheta^2 + \sin\vartheta^2 = 1$$

$$h(x) = h_0 + R\left(1 - \sqrt{1 - \frac{x^2}{R^2}}\right) = h_0 + R - \sqrt{R^2 - x^2}$$

$$\frac{\partial h}{\partial x} = \frac{x}{\sqrt{R^2 - x^2}}$$



1D Reynolds equation:

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial x}\left(h\frac{u_a + u_b}{2}\right) = 0$$

$$3h^2\frac{\partial h}{\partial x}\frac{\partial p}{\partial x} + h^3\frac{\partial^2 p}{\partial x^2} = 12\mu\frac{\partial h}{\partial x}\cdot\frac{u_a + u_b}{2} + 12\mu\frac{h}{2}\frac{\partial u_a}{\partial x} + 12\mu\cancel{\frac{h}{2}\frac{\partial u_b}{\partial x}} \longrightarrow c\frac{\partial p}{\partial x} + a\frac{\partial^2 p}{\partial x^2} = e$$

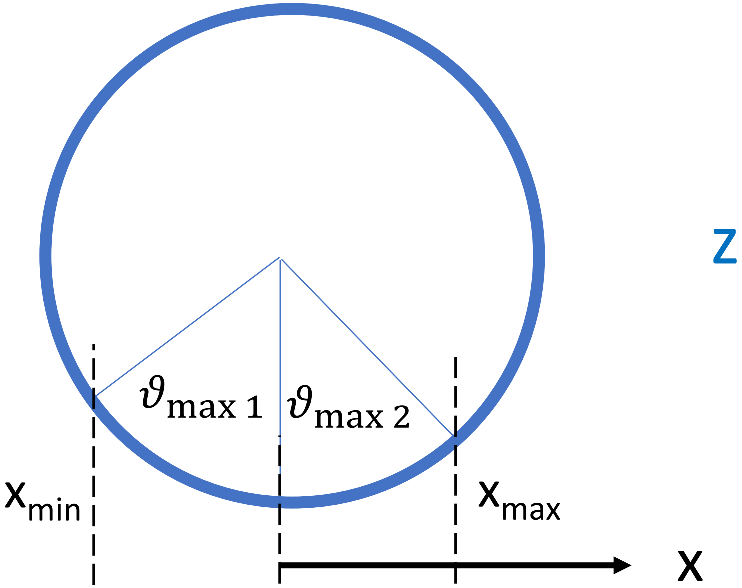
SOLUTION OF SIMPLE CASES

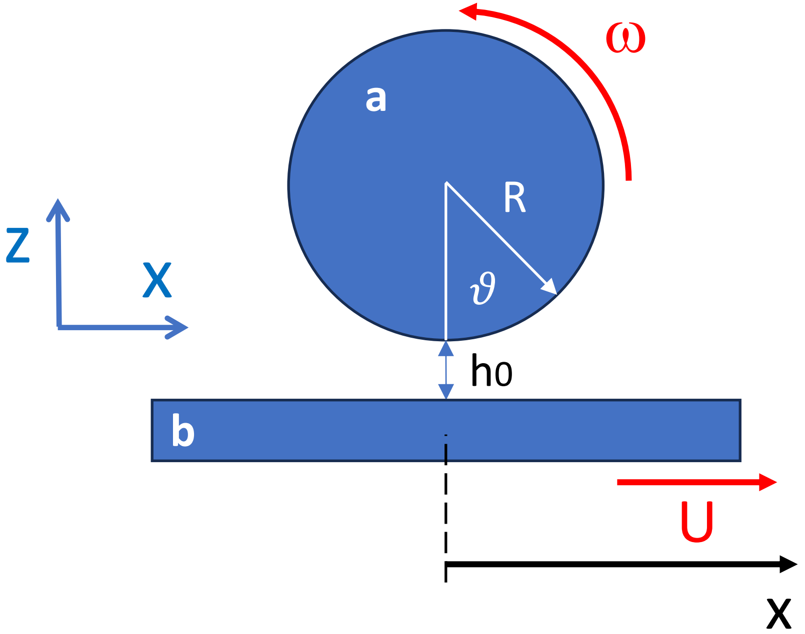
LUBRICATED CYLINDER ON PLANE

Angles $\vartheta_{\max 1}$ and $\vartheta_{\max 2}$ can be defined for the boundary conditions:

$$x_{min} = -R \sin \vartheta_{\max 1}$$

$$x_{max} = R \sin \vartheta_{\max 2}$$





BCs:

$$u_a = \omega R \cos \vartheta = \omega \sqrt{R^2 - x^2}$$

$$u_b = U$$

$$p(x_{min}) = p_a$$
$$p(x_{max}) = p_a$$

$$\frac{\partial u_a}{\partial x} = -\frac{\omega x}{\sqrt{R^2 - x^2}}$$

Component of speed of the upper cylinder along the x-direction

SOLUTION OF SIMPLE CASES

LUBRICATED CYLINDER ON PLANE

$$c\frac{\partial p}{\partial x}+a\frac{\partial^2 p}{\partial x^2}=e$$

Discrete form:

$$c_i\frac{p_{i+1}-p_{i-1}}{2\Delta x}+a_i\frac{p_{i+1}-2p_i+p_{i-1}}{\Delta x^2}=e_i$$

$$p_{i-1}\left(\frac{a_i}{\Delta x^2}-\frac{c_i}{2\Delta x}\right)-2p_i\frac{a_i}{\Delta x^2}+p_{i+1}\left(\frac{a_i}{\Delta x^2}+\frac{c_i}{2\Delta x}\right)=e_i$$

Coefficients read as follows:

$$a_i=h_i^3$$

$$c_i=3h_i^2\left.\frac{\Delta h}{\Delta x}\right|_i=3h_i^2\frac{h_{i+\frac{1}{2}}-h_{i-\frac{1}{2}}}{\Delta x}$$

$$=3h_i^2\frac{x_i}{\sqrt{R^2-x_i^2}}$$

$$e_i=12\mu\left.\left(\frac{u_a+u_b}{2}\frac{\partial h}{\partial x}+\frac{h}{2}\frac{\partial u_a}{\partial x}\right)\right|_i=12\mu\left(\frac{U+\omega R\cos\vartheta}{2}\frac{x}{\sqrt{R^2-x^2}}-\frac{h}{2}\frac{\omega x}{\sqrt{R^2-x^2}}\right)$$

$$=12\mu\left(\frac{U+\omega\sqrt{R^2-x^2}}{2}\frac{x}{\sqrt{R^2-x^2}}-\frac{h_0+R-\sqrt{R^2-x^2}}{2}\frac{\omega x}{\sqrt{R^2-x^2}}\right)=$$

$$=12\mu\left(\frac{Ux}{2\sqrt{R^2-x^2}}+\frac{\omega x}{2}-\frac{h_0+R}{2}\frac{\omega x}{\sqrt{R^2-x^2}}+\frac{\omega x}{2}\right)\quad=\quad12\mu\frac{U-(h_0+R)\omega}{2\sqrt{R^2-x_i^2}}x_i+\omega x_i$$

SOLUTION OF SIMPLE CASES

LUBRICATED CYLINDER ON PLANE

Let us find a solution in order to **avoid negative relative pressure** → *cavitation does not appear by hypothesis*

BCs:

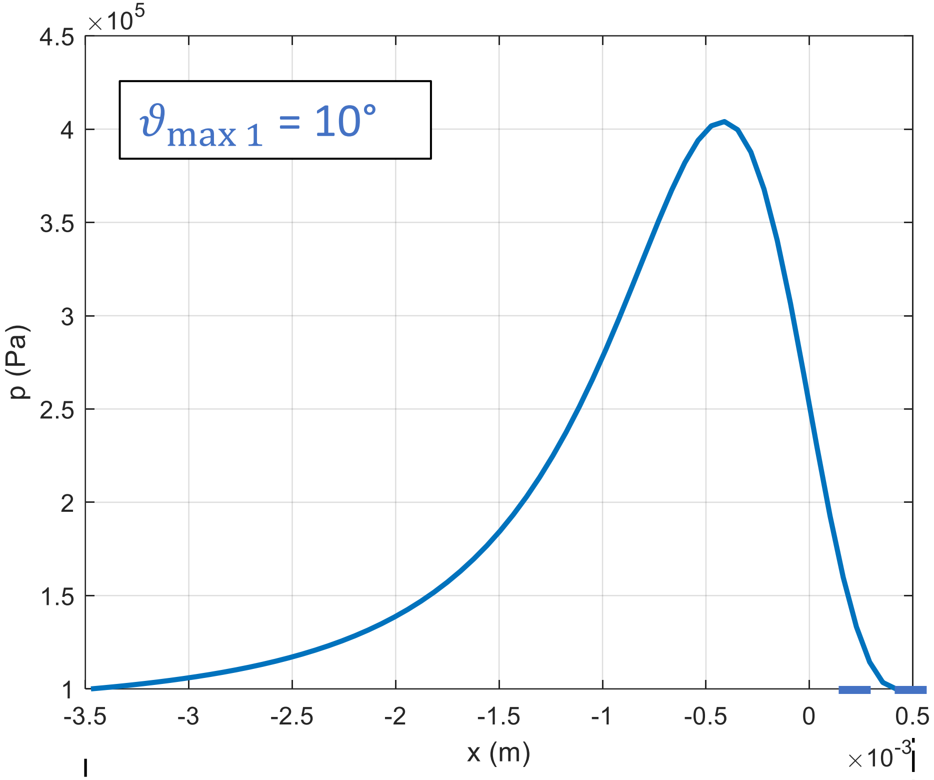
$$p(x_{min} =?) = p_a$$

$$p(x_{max} =?) = p_a$$

$$\left.\frac{\partial p}{\partial x}\right|_{x_{\max\ 2}=?} = 0$$

Let us determine *suitably tuned boundaries of the domain* with consistent BCs

If $\vartheta_{\max\ 1}$ (i.e. x_{max}) is given one can look for $\vartheta_{\max\ 2}$ *iteratively,* in order to get $p(\vartheta_{\max\ 2}) = p_a$ and $\left.\frac{\partial p}{\partial x}\right|_{\vartheta_{\max\ 2}} = 0$ at the output section.



$$\left.\frac{\partial p}{\partial x}\right|_{\vartheta_{\max\ 2}} = 0$$

x_{min}

x_{max}

Iterations provide:

$\vartheta_{\max\ 2}$ = 1.2°

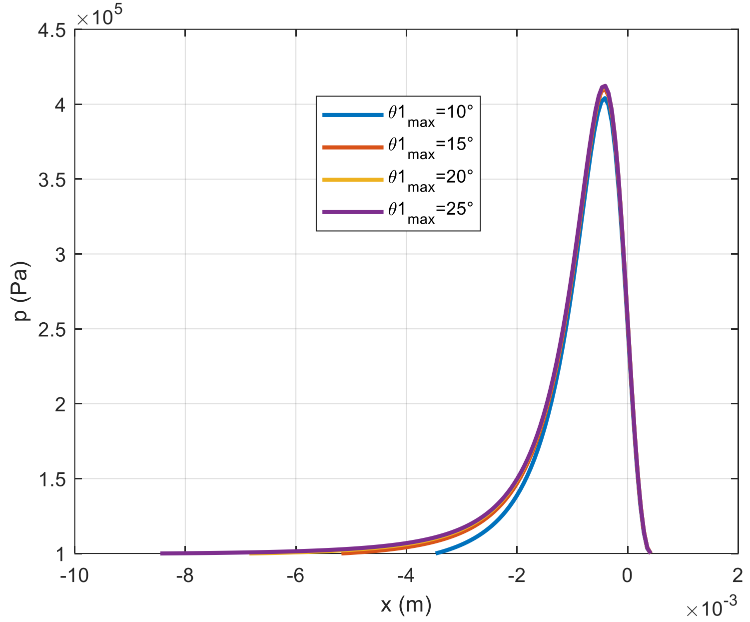
Example with:

W = 0.1 mm/s
H = 20 mm
R = 20 mm
U = 1 m/s
μ = 0.15 Pa·s
Ω = 1000 rpm

SOLUTION OF SIMPLE CASES

LUBRICATED CYLINDER ON PLANE

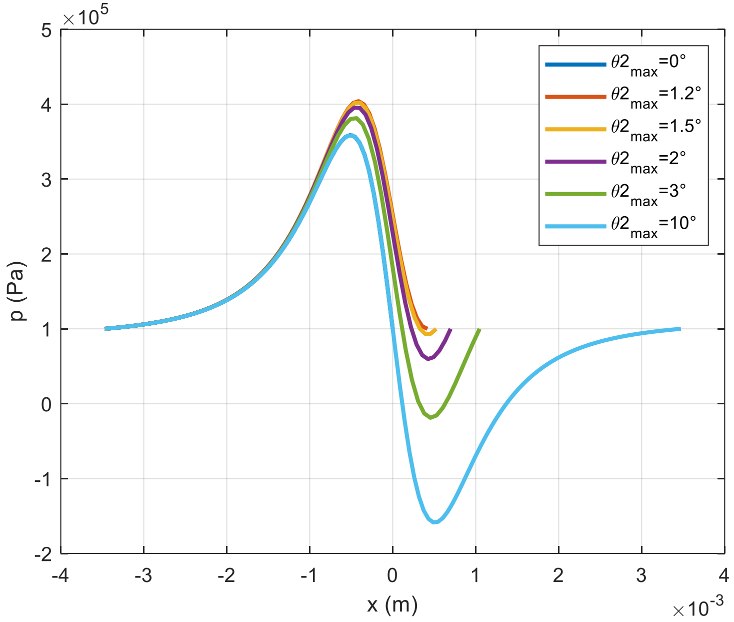
Results are compared changing $\vartheta_{\max 1}$, with constant $\vartheta_{\max 2} = 1.2°$



NB: it is recommended not to exceed $\vartheta_{\max 1} = 10°$ as the hypothesis of Reynolds equation is thin film, i.e. ***h/R<<1***

When $\vartheta_{\max 1}$ is increased, the pressure distribution changes just a little, as it is very low when *h* is large

Results are compared changing $\vartheta_{\max 2}$, with constant $\vartheta_{\max 1} = 10°$



Notice that the solution with $\vartheta_{\max 1} = \vartheta_{\max 2}$ is antisymmetrical and the first part coincide with the solution of $\vartheta_{\max 2} = 0°$

SOLUTION OF SIMPLE CASES

ASSIGNEMENT B

IMPLEMENT A MATLAB SCRIPT TO SOLVE THE STEADY STATE SOLUTION AND OBTAINS THE PRESSURE DISTRIBUTION INTO THE GAP.

THE SOLUTION HAS TO BE FOUND ITERATIVELY BY ITERATING ON THE INLET AND OUTLET ANGLES

```
%% Clear workspace and
variables
clear all
close all
clc
```

```
%% Preprocessing
%Introducing variables
pa = 1e5;          %Pa
mu = 150e-3;       %Pa*s viscosity
h0 = 20e-6;        %m minimum gap
R = 20e-3;         %m
U=1;               %m/s
omega = -10*U/R;       %rad/s
N = ...;           %n. of internal nodes
tetamax1 = ...;
xmin = -...;
```

```
% Iterative discretization
tetamax2 = ...;
xmax = ...;
x = ...
x_r = ...; %internal nodes
deltax = ...
h = ...;     %N elements
dhdx = ;   %N elements
...

%Iterative solver
...
```

SOLUTION OF SIMPLE CASES

Analytical & numerical solution of simple cases

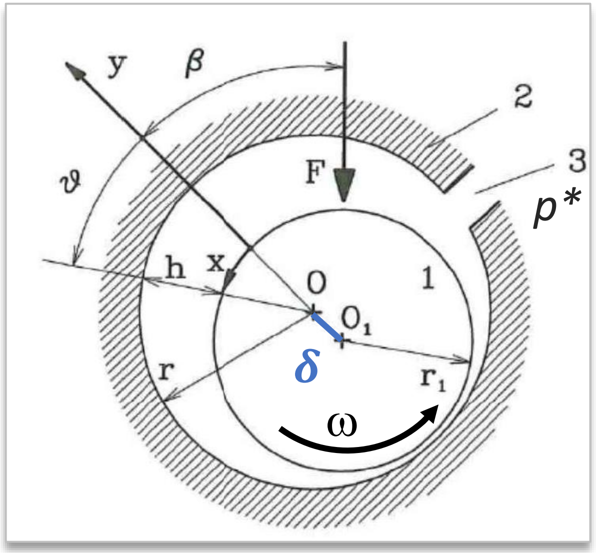
- 1D hydrodynamic slider
- 1D slider with normal squeeze motion
- 1D slider, hydrostatic effect

Numerical solution of simple cases

- Lubricated cylinder on plane
- Journal bearing with infinite length
- 2D hydrodynamic slider

SOLUTION OF SIMPLE CASES

INFINITE LENGTH JOURNAL BEARING



1. *Shaft*
2. *Bushing*
3. *Oil supply port*

R is 0.1% larger than r1

$OO_1 = \delta$: eccentricity

y: line of centers

β: attitude angle

h_0: mean radial gap

$\varepsilon = \delta / h_0$: eccentricity ratio

Incompressible fluid (ρ is constant)
No cavitation

BCs: $p(\vartheta = \alpha) = p^*$

At the lubricant supply port pressure p* is assumed to be known

NB: there exists the analytical solution!

Film thickness in polar coordinates: $h(\theta) = h_0 + \delta \cos\vartheta = h_0(1 + \varepsilon \cos\vartheta)$

1D RE in polar coordinates (r, ϑ):

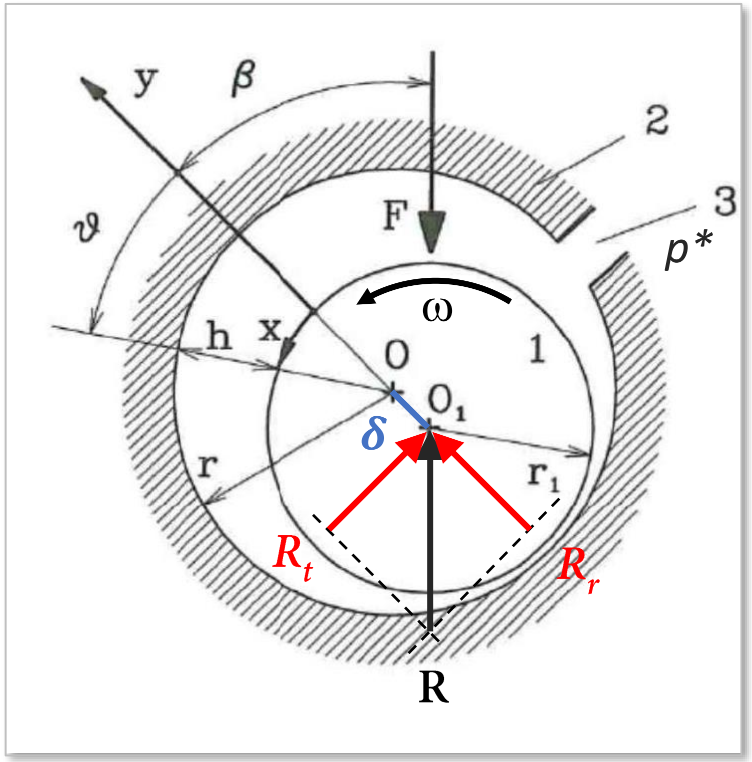
$$\frac{\partial}{r\partial\vartheta}\left(\frac{h^3}{12\mu}\frac{\partial p}{r\partial\vartheta} + \frac{\omega r}{2}h\right) = 0$$

The following hypotheses hold:

- The bushing fixed (u_b=0, w_b=0)
- The center of the journal fixed (w_a=0, u_a= ωr)

SOLUTION OF SIMPLE CASES

INFINITE LENGTH JOURNAL BEARING



Radial (along the line of actions) and tangential (normal to the line of actions) bearing reaction components due to the pressure distribution:

$$\begin{bmatrix} R_r \\ R_t \end{bmatrix} = \iint p \begin{bmatrix} \cos\vartheta \\ \sin\vartheta \end{bmatrix} r d\vartheta dz$$

→ *The effect of tangential stress due to friction is neglected*

In steady state conditions there is equilibrium between the external load *F* applied on the shaft and the reaction force *R* exerted by the bearing.

Since the journal has infinite length, we can calculate the force *f* per unit of axial length:

$$\begin{bmatrix} f_r \\ f_t \end{bmatrix} = \int_0^{2\pi} p \begin{bmatrix} \cos\vartheta \\ \sin\vartheta \end{bmatrix} r d\vartheta$$

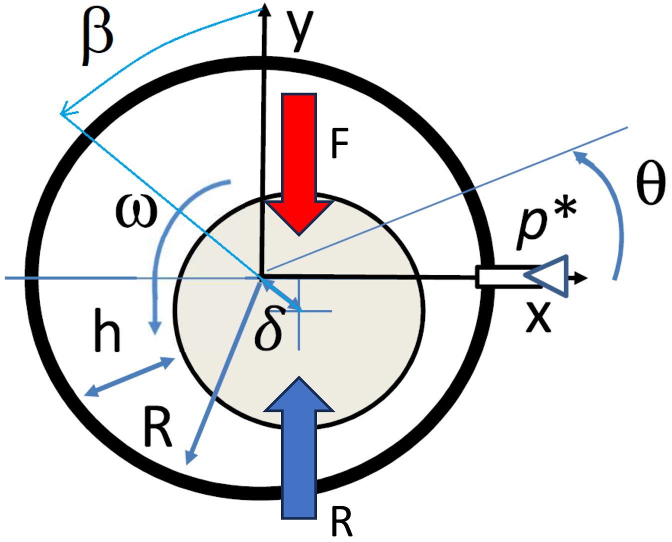
INFINITE LENGTH JOURNAL BEARING, NUMERICAL SOLUTION

In order to simplify the numerical solution, without loss of generality we define a **fixed reference system Oxy** and assume the oil supply port to be aligned with horizontal axis x.

The external force is vertical. The attitude angle β and the eccentricity δ are to be determined to have the equilibrium between the external force F and the journal reaction R.

$$h(\vartheta) = h_0 + \delta \cos(\vartheta - \beta - \pi/2)$$

1. *Iteration for module of R* → given an attempt attitude angle β, iterate the value of eccentricity δ increasing it from a small value until the module of *R* coincides with *F*
2. *Iteration for direction of R* → The attitude angle β is then changed and point 1 is to be repeated to find the new eccentricity with the new angle
3. The final attitude angle is found when F is aligned with R.

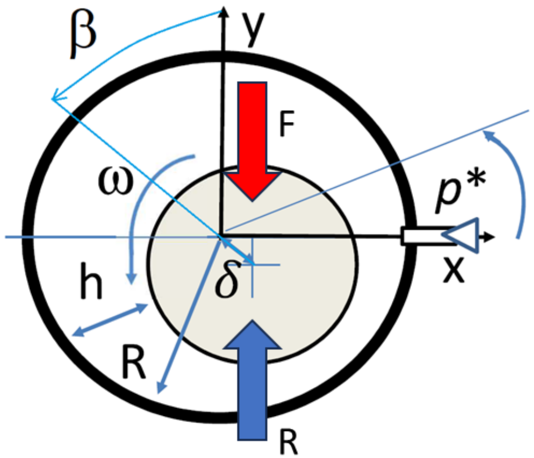


Force (per unit of length) components along x and y:

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \int_0^{2\pi} p \begin{bmatrix} \cos\vartheta \\ \sin\vartheta \end{bmatrix} r d\vartheta$$

SOLUTION OF SIMPLE CASES

INFINITE LENGTH JOURNAL BEARING, NUMERICAL SOLUTION



The problem is similar to the 1D hydrodynamic slider.

$$\frac{\partial}{r\partial\vartheta}\left(\frac{h^3}{12\mu}\frac{\partial p}{r\partial\vartheta}+\frac{\omega r}{2}h\right)=0$$

BCs: $p(\vartheta=0)=p(\vartheta=2\pi)=p^*$

p^* large enough to avoid cavitation

Discretization:

$$\frac{1}{12\mu}\frac{p_{i+1}-p_i}{r\Delta\vartheta}h^3_{i+\frac{1}{2}}+\frac{\omega r}{2}h_{i+\frac{1}{2}}=\frac{1}{12\mu}\frac{p_i-p_{i-1}}{r\Delta\vartheta}h^3_{i-\frac{1}{2}}+\frac{\omega r}{2}h_{i-\frac{1}{2}}$$

There result *N* linear algebraic equations in *N* unknown variables

SOLUTION OF SIMPLE CASES

ASSIGNEMENT C

IMPLEMENT A MATLAB SCRIPT TO SOLVE THE STEADY STATE SOLUTION AND OBTAINS THE PRESSURE DISTRIBUTION INTO THE JOURNAL BEARING.

THE SOLUTION HAS TO BE FOUND ITERATIVELY BY ITERATING ON ATTITUDE ANGLE AND ECCENTRICITY

MATLAB

```
%% Clear workspace and variables
clear all
close all
clc
```

```
%% Preprocessing
%Introducing variables
pa = 1e5;           %Pa ambient pressure
pstar = 20e5;       %Pa inlet pressure
mu = 150e-3;        % Pa*s
h0 = 200e-6;        % symmetric gap thickness
omega = 1000*2*pi/60;    %rad/s
R=30e-3;                 %m
epsx = ...;
epsy = ...;
eps = ...;
fi = atan2(epsy,epsx);   %rad = beta + pi/2
N = ...;
```

```
%Discretization
teta =...;
dteta = ...;
gap = ...;
h = ...; % internal nodes

%Solver
...
```

SOLUTION OF SIMPLE CASES

Analytical & numerical solution of simple cases

- 1D hydrodynamic slider
- 1D slider with normal squeeze motion
- 1D slider, hydrostatic effect

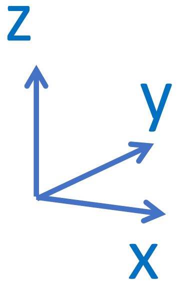
Numerical solution of simple cases

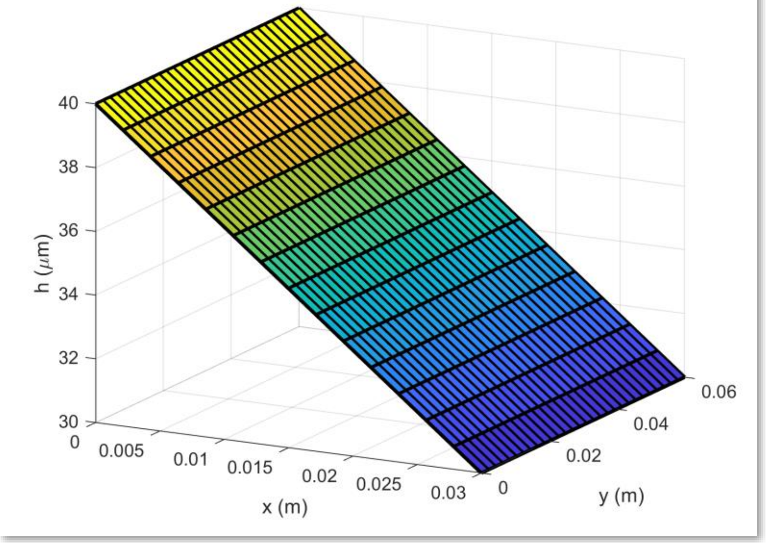
- Lubricated cylinder on plane
- Journal bearing with infinite length
- 2D hydrodynamic slider

SOLUTION OF SIMPLE CASES

FINITE LENGTH SLIDER, NUMERICAL SOLUTION

- Suppose a convergent gap along *x* and constant speed *U* along the x-direction.
- The speed *V* along the y-direction and W along the z-direction is supposed to be null and the gap is independent of the y-coordinate.





$$h(x,y) = h(x) \;=\; h_{max} - \frac{h_{max} - h_{min}}{L_x} x$$

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{12\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\rho h^3}{12\mu}\frac{\partial p}{\partial y}\right) - \frac{\partial}{\partial x}\left(\rho h \frac{u_a + \cancel{u_b}}{2}\right) \cancel{- \frac{\partial}{\partial y}\left(\rho h \frac{v_a + v_b}{2}\right)} \cancel{- \rho\left(w_a - w_b - u_a\frac{\partial h}{\partial x} - v_a \frac{\partial h}{\partial y}\right)} - h\frac{\partial \cancel{\rho}}{\partial t} = 0$$

BCs:

$u_a = 0$
$u_b = -U$
$v_a = 0$
$v_b = 0$

$w_a = 0$
$w_b = 0$

$p(0,y) = p_a$
$p(L_x,y) = p_a$
$p(x,0) = p_a$
$p(x,L_y) = p_a$

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\right) - \frac{U}{2}\frac{\partial h}{\partial x} = 0$$

Incompressible fluid (ρ is constant)

SOLUTION OF SIMPLE CASES

FINITE LENGTH SLIDER, NUMERICAL SOLUTION

Let us start from the most general case with terms also along the y-direction:

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right)-6\mu U\frac{\partial h}{\partial x}-6\mu V\frac{\partial h}{\partial y}=0$$

Discretization of 2D RE

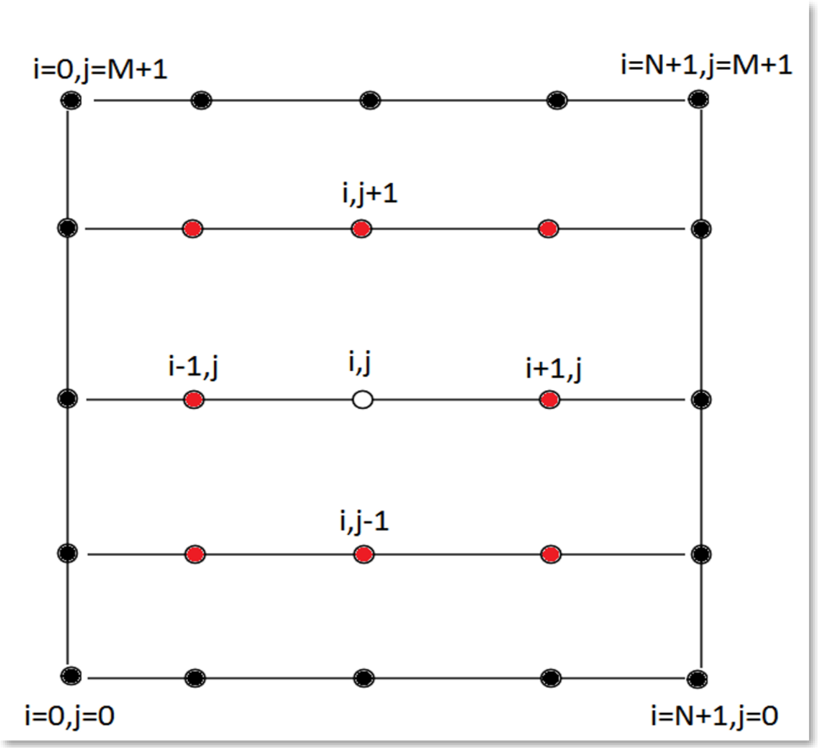
N+2 nodes along the x-direction, from *i = 0* to *i = N+1*

M+2 nodes along the y-direction, from *j = 0* to *j = M+1*

Finite difference discretization: central formulation

$$\frac{dp}{dx}=\frac{p_{i+1}-p_{i-1}}{2\Delta x}$$

$$\frac{d^2p}{dx^2}=\frac{p_{i+1}-2p_i+p_{i-1}}{\Delta x^2}$$



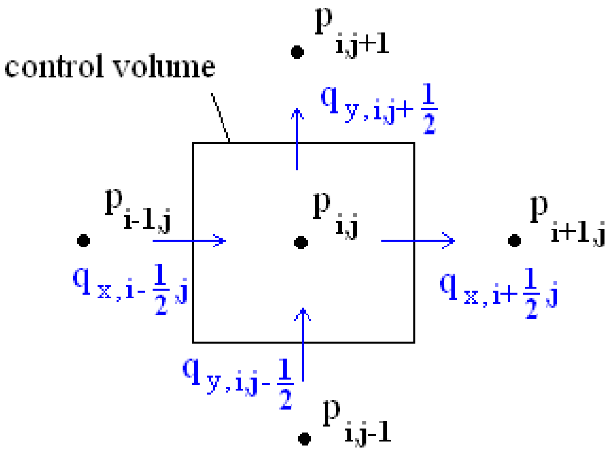
SOLUTION OF SIMPLE CASES

FINITE LENGTH SLIDER, NUMERICAL SOLUTION

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}-\frac{U}{2}h\right)+\frac{\partial}{\partial y}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial y}-\frac{V}{2}h\right)=0$$

Volume flow balance along the x-direction and y-direction

5 points stencil



$$\underbrace{\left.\left(\frac{1}{12\mu}\frac{dp}{dx}h^3-\frac{Uh}{2}\right)\right|_{i+\frac{1}{2},j}}_{q^x_{i+\frac{1}{2},j}}\underbrace{-\left.\left(\frac{1}{12\mu}\frac{dp}{dx}h^3-\frac{Uh}{2}\right)\right|_{i-\frac{1}{2},j}}_{q^x_{i-\frac{1}{2},j}}\underbrace{+\left.\left(\frac{1}{12\mu}\frac{dp}{dy}h^3-\frac{Vh}{2}\right)\right|_{i,j+\frac{1}{2}}}_{q^y_{i,j+\frac{1}{2}}}\underbrace{-\left.\left(\frac{1}{12\mu}\frac{dp}{dy}h^3-\frac{Vh}{2}\right)\right|_{i,j-\frac{1}{2}}}_{q^y_{i,j-\frac{1}{2}}}=0$$

$$\frac{-U}{2}h_{i+\frac{1}{2}}+\frac{1}{12\mu}\frac{p_{i+1}-p_i}{\Delta x}h^3_{i+\frac{1}{2}}+\frac{U}{2}h_{i-\frac{1}{2}}-\frac{1}{12\mu}\frac{p_i-p_{i-1}}{\Delta x}h^3_{i-\frac{1}{2}}-\frac{V}{2}h_{j+\frac{1}{2}}+\frac{1}{12\mu}\frac{p_{j+1}-p_j}{\Delta y}h^3_{j+\frac{1}{2}}+\frac{V}{2}h_{j-\frac{1}{2}}-\frac{1}{12\mu}\frac{p_j-p_{j-1}}{\Delta y}h^3_{j-\frac{1}{2}}=0$$

SOLUTION OF SIMPLE CASES

FINITE LENGTH SLIDER, NUMERICAL SOLUTION

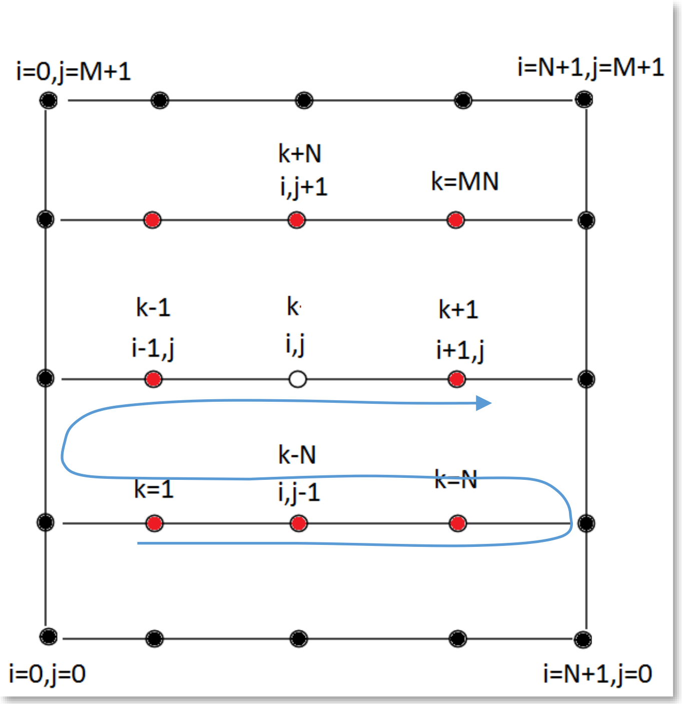
The equations can be written in a system using the **lexicographic order**; index *k* refers to the *k-equation*:

$$k = i + (j-1)N \qquad k \in I[1,(N)(M)]$$

$$\frac{1}{12\mu}\frac{h^3_{i,j-\frac{1}{2}}}{\Delta y}p_{i,j-1} + \frac{1}{12\mu}\frac{h^3_{i-\frac{1}{2},j}}{\Delta x}p_{i-1,j} - \frac{1}{12\mu}\left(\frac{h^3_{i+\frac{1}{2},j}}{\Delta x}+\frac{h^3_{i-\frac{1}{2},j}}{\Delta x}+\frac{h^3_{i,j+\frac{1}{2}}}{\Delta y}+\frac{h^3_{i,j-\frac{1}{2}}}{\Delta y}\right)p_{i,j}$$

$$+\frac{1}{12\mu}\frac{h^3_{i+\frac{1}{2},j}}{\Delta x}p_{i+1,j}+\frac{1}{12\mu}\frac{h^3_{i,j+\frac{1}{2}}}{\Delta y}p_{i,j+1} = \frac{U}{2}\left(h_{i+\frac{1}{2},j}-h_{i-\frac{1}{2},j}\right)\cancel{+\frac{V}{2}\left(h_{i,j+\frac{1}{2}}-h_{i,j-\frac{1}{2}}\right)}$$

$$A_{k,k-N}p_{k-N} + A_{k,k-1}p_{k-1} + A_{k,k}p_k + A_{k,k+1}p_{k+1} + A_{k,k+N}p_{k+N} = b_k$$



$$\frac{1}{12\mu}\frac{h^3_{k-N}}{\Delta y}p_{k-N} + \frac{1}{12\mu}\frac{h^3_{k-1}}{\Delta x}p_{k-1} - \frac{1}{12\mu}\left(\frac{h^3_{k+1}}{\Delta x}+\frac{h^3_{k-1}}{\Delta x}+\frac{h^3_{k+N}}{\Delta y}+\frac{h^3_{k-N}}{\Delta y}\right)p_k$$

$$+\frac{1}{12\mu}\frac{h^3_{k+1}}{\Delta x}p_{k+1}+\frac{1}{12\mu}\frac{h^3_{k+N}}{\Delta y}p_{k+N} = \frac{U}{2}\left(h_{k+1}-h_{k-1}\right)$$

SOLUTION OF SIMPLE CASES

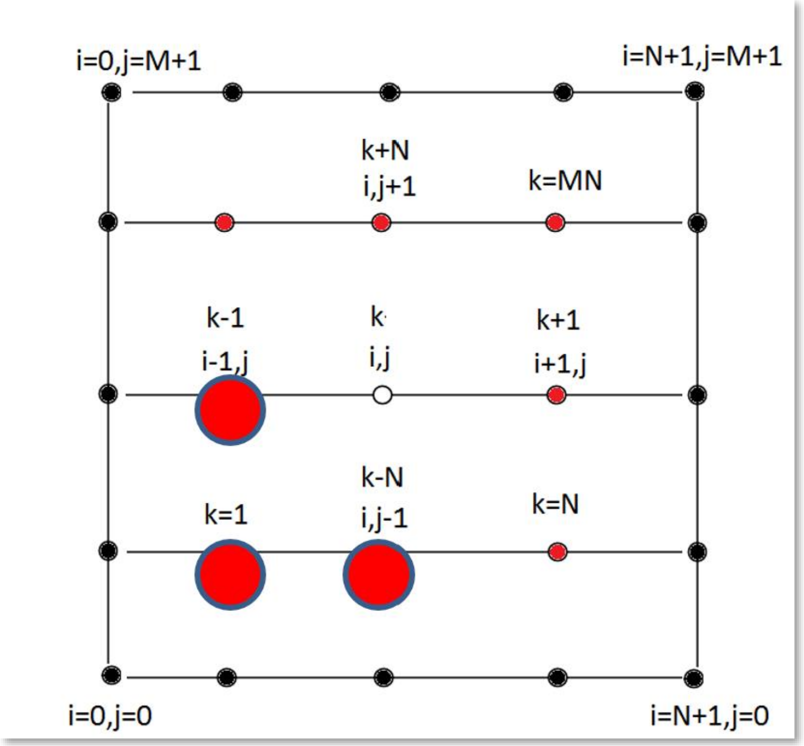
FINITE LENGTH SLIDER, NUMERICAL SOLUTION

The nodes adjacent to the borders with Dirichlet boundary conditions do not use all the stencil points as the pressure is known at the borders.

For instance, in the case ***i* = 1, *j* = 1 → *k* = 1** just three adjacent have unknown pressure level:

$$\frac{1}{12\mu}\frac{h_{k-N}^3}{\Delta y}p_a + \frac{1}{12\mu}\frac{h_{k-1}^3}{\Delta x}p_a - \frac{1}{12\mu}\left(\frac{h_{k+1}^3}{\Delta x} + \frac{h_{k-1}^3}{\Delta x} + \frac{h_{k+N}^3}{\Delta y} + \frac{h_{k-N}^3}{\Delta y}\right)p_k$$
$$+ \frac{1}{12\mu}\frac{h_{k+1}^3}{\Delta x}p_{k+1} + \frac{1}{12\mu}\frac{h_{k+N}^3}{\Delta y}p_{k+N} = \frac{U}{2}(h_{k+1} - h_{k-1})$$

The terms in orange will move in the vector of known terms b_k



SOLUTION OF SIMPLE CASES

FINITE LENGTH SLIDER, NUMERICAL SOLUTION

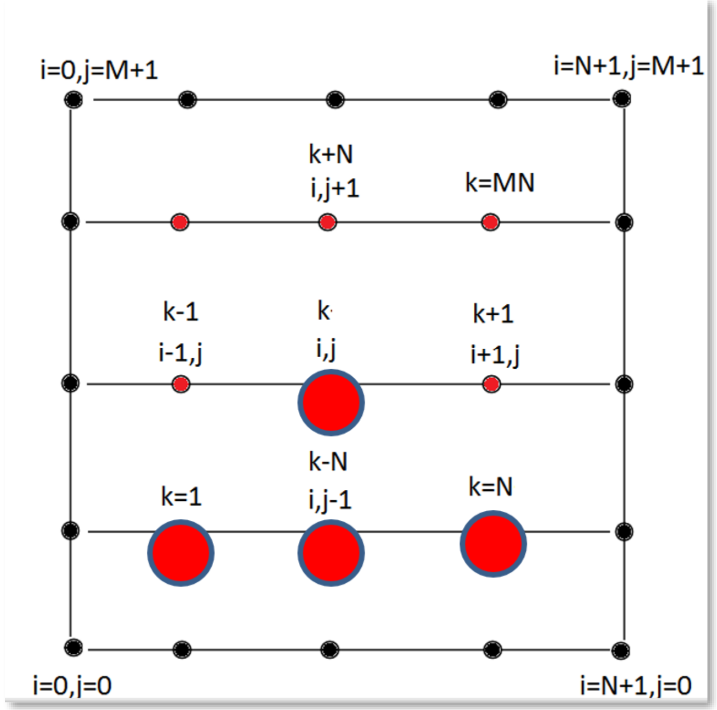
The nodes adjacent to the borders with Dirichlet boundary conditions do not use all the stencil points as the pressure is known at the borders.

For instance, in the case ***i* = 2, *j* = *N*-1** → ***k* = 2 to *N*-1** just three adjacent have unknown pressure level:

$$\frac{1}{12\mu}\frac{h_{k-N}^3}{\Delta y}p_a + \frac{1}{12\mu}\frac{h_{k-1}^3}{\Delta x}p_{k-1} - \frac{1}{12\mu}\left(\frac{h_{k+1}^3}{\Delta x} + \frac{h_{k-1}^3}{\Delta x} + \frac{h_{k+N}^3}{\Delta y} + \frac{h_{k-N}^3}{\Delta y}\right)p_k$$

$$+ \frac{1}{12\mu}\frac{h_{k+1}^3}{\Delta x}p_{k+1} + \frac{1}{12\mu}\frac{h_{k+N}^3}{\Delta y}p_{k+N} = \frac{U}{2}(h_{k+1} - h_{k-1})$$

The terms in orange will move in the vector of known terms b_k



SOLUTION OF SIMPLE CASES

FINITE LENGTH SLIDER, NUMERICAL SOLUTION

When all the equations are written, it results a linear system with **pentadiagonal matrix:**

$$[A]\{p\} = \{b\}$$

$$A_{k,k-N}p_{k-N} + A_{k,k-1}p_{k-1} + A_{k,k}p_k + A_{k,k+1}p_{k+1} + A_{k,k+N}p_{k+N} = b_k$$

- $[A]$ is a square matrix of size ***N·M x N·M***; it can be described as a blocks tridiagonal matrix, with tridiagonal submatrix ***D*** and diagonal submatrix ***C***

- *$\{p\}$ and $\{b\}$ are column vectors of size **N·M***

$$A = \begin{pmatrix} D & C & & & \\ C & D & C & & \\ & C & D & C & \\ & & C & D & C \\ & & & C & D \end{pmatrix}$$

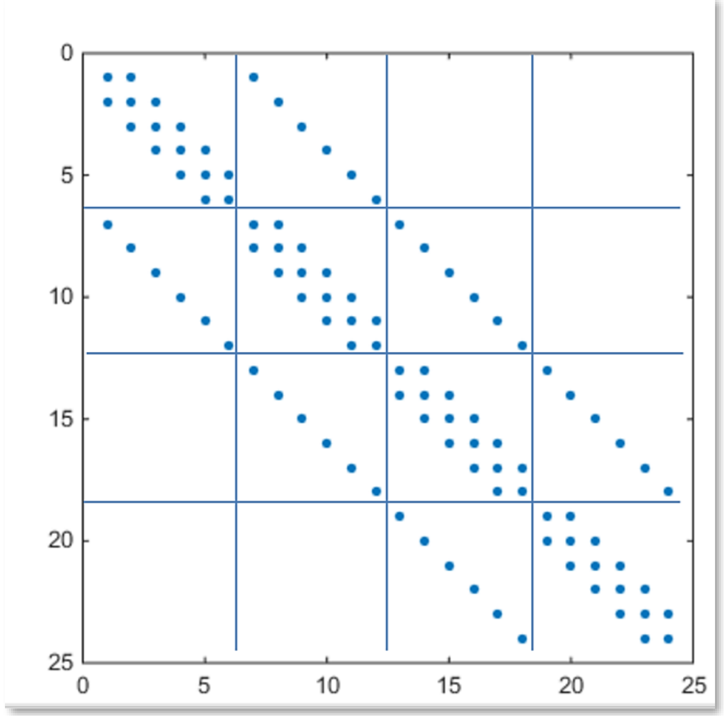
j = 1

j = 2 to M-1

j = M

Where:

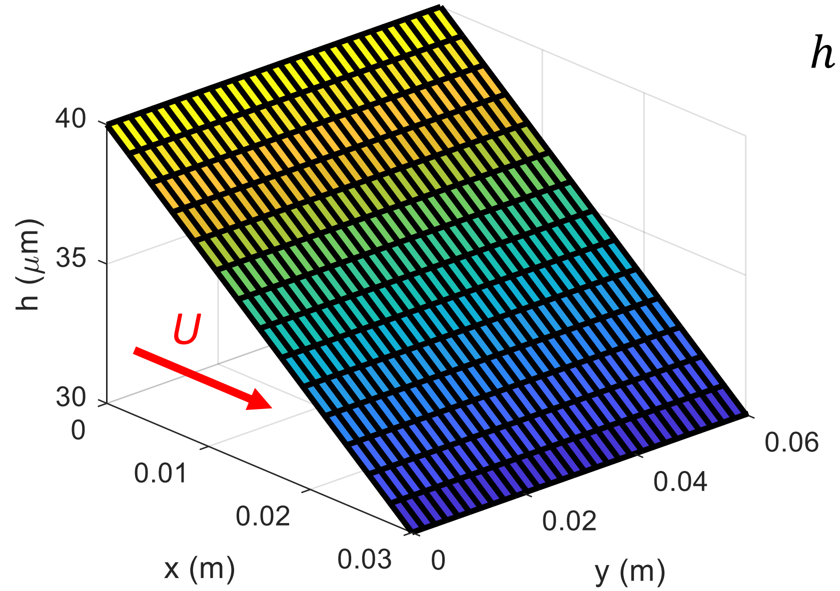
Size(D) = Size(C) = ***N x N***



Since h(x, y) = h(x) matrices D and C are everywhere equal to each

SOLUTION OF SIMPLE CASES

FINITE LENGTH SLIDER, NUMERICAL SOLUTION



$$h(x,y) = h_{max} - \frac{h_{max} - h_{min}}{L_x} x$$

Example :

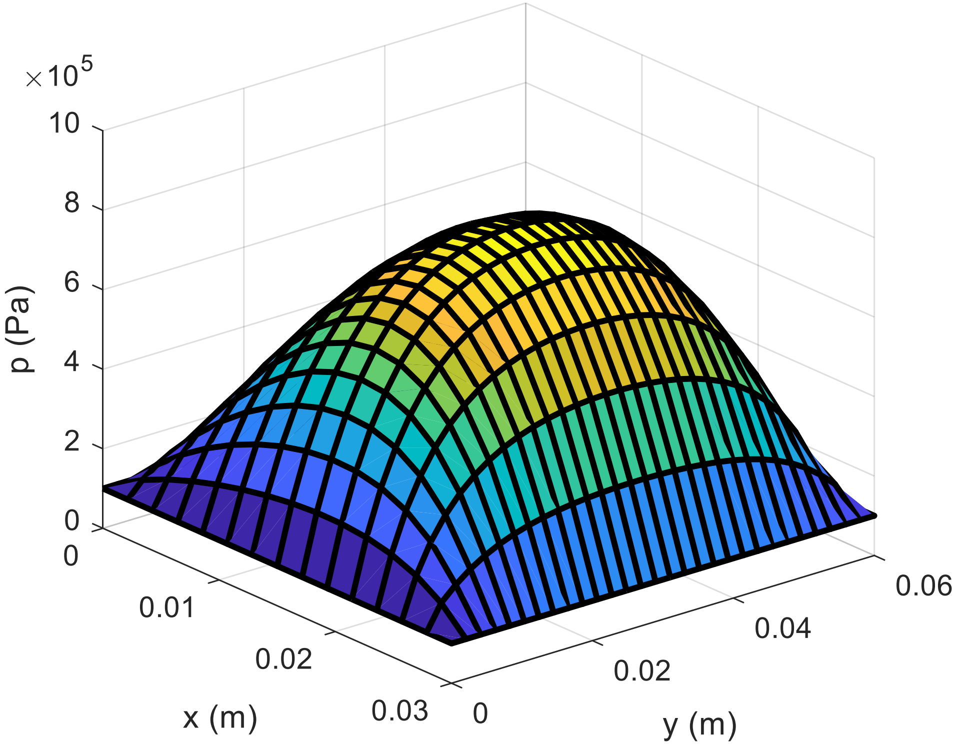
$h_{min} = 30\mu m$

$h_{max} = 40\mu m$

$Lx = 30mm$
$Ly = 60mm$

$U = 1m/s, V = 0$

$\mu = 0.15 Pa \cdot s$



SOLUTION OF SIMPLE CASES

ASSIGNMENT D

IMPLEMENT A MATLAB SCRIPT TO SOLVE THE STEADY STATE 2D SOLUTION AND OBTAINS THE PRESSURE DISTRIBUTION INTO THE GAP.

```
%% Clear workspace and variables
clear all
close all
clc
```

```
%% Preprocessing
%Introducing variables
pa = 1e5;          %Pa
mu = 150e-3;       %Pa*s viscosity
hmin = 20e-6;      %m
hmax = 40e-6;      %m
Lx = 30e-3;        %m
Ly = 60e-3;        %m
U = 1;             %m/s
N = ...;           %n. of internal nodes
M = ...;           %n. of internal nodes
```

```
%Discretization
x = ...;
deltax =...;
y =...;
deltay =...;
[X,Y] = meshgrid(x,y);
X = X';
Y = Y';
gap =...;
h = ...;     %internal nodes
```

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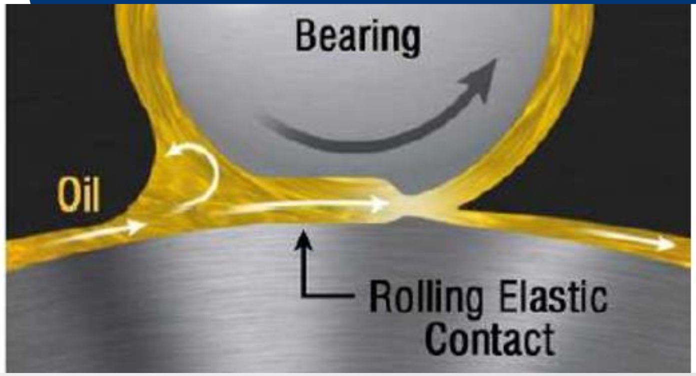
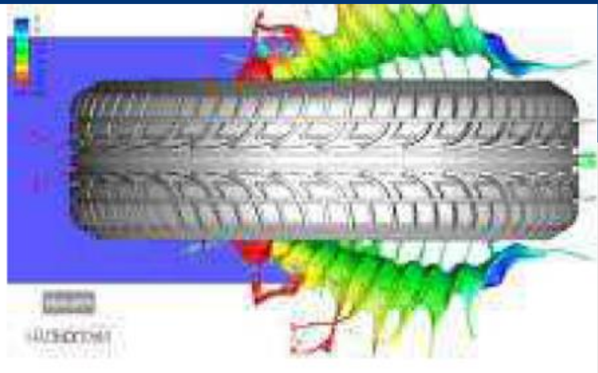
e Aerospaziale





NUMERICAL MODELS FOR LUBRICATION

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